HW #2 Math 265B, L. Vese
Due on Friday, October 24

[1] Let V be a complex Hilbert space, $a : V \times V \to C$ a sesquilinear form, $L : V \to C$ an anti-linear form. Assume that a is Hermitian, thus $a(v,u) = \overline{a(u,v)}$, $\forall u, v \in V$, and that $a(v,v) \geq 0$. Consider the problems

$$\text{(V) Find } u \in V \text{ s.t. } a(u,v) = L(v), \ \forall v \in V,$$

$$\text{(M) Find } u \in V \text{ s.t. } J(u) = \inf_{v \in V} J(v),$$

with $J : V \to R$ defined by $J(v) = \frac{1}{2}a(v,v) - \text{Re}L(v)$.
Show that $u \in V$ is solution of (V) iff $u \in V$ is solution of (M).

[2] Consider the inhomogeneous Neumann problem

$$u - \triangle u = f \text{ in } \Omega, \qquad \frac{\partial u}{\partial \vec{n}} = g \text{ on } \partial\Omega,$$

where $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$, with Ω an open and bounded subset of R^2.
(i) Derive the weak variational formulation, denoted by (V), using test functions $v \in V = H^1(\Omega)$, which is a Hilbert space.
(ii) Write the associated minimization problem, denoted by (M).
(iii) Show directly, without applying the general result, that (V) and (M) are equivalent.

[3] Let V be a normed vector space, and V^* its dual. Show that:
(i) $|\|u\| - \|v\|| \leq \|u - v\|$
(ii) if $u_n \to u$, then $\|u_n\| \to \|u\|$ as $n \to \infty$.
(iii) If $u_n \to u$ in V, then $u_n \rightharpoonup u$ weakly in V.

[4] Let V be a complex Hilbert space, K a closed, convex, non-empty subset of V, and $u_0 \in V$. Derive the optimality condition for the projection $\bar{u} = Proj_K u_0$ in this complex case.