HW #1 Math 265B, Fall 2008, L. Vese
Due on Friday, October 10 (tentative date)

[**1**] Show that the Lax-Milgram theorem contains as particular case the Riesz representation theorem (in other words, assume Lax-Milgram and prove the Riesz representation theorem).

[**2**] Let V be a Hilbert space, and $a : V \times V \to R$ be a bilinear form. Show that: a bounded is equivalent with a continuous.

[**3**] Consider the following minimization problem proposed by Weierstrass (apparently close to the Dirichlet problem):

$$\min\Big\{\int_{-1}^{+1} x^2\Big(\frac{du}{dx}\Big)^2 dx : \quad u(-1) = a, \quad u(+1) = b\Big\},$$

where $a \neq b$. We wish to show that this problem cannot have a regular (continuous, piecewise C^1) solution u that satisfies the boundary condition.

(a) Consider the approximation problem

$$\min\Big\{\int_{-1}^{+1} \Big(x^2 + \frac{1}{n^2}\Big)\Big(\frac{du}{dx}\Big)^2 dx : \quad u(-1) = a, \quad u(+1) = b\Big\}.$$

Show that $u_n(x) = \frac{a+b}{2} - \frac{a-b}{2}\frac{\arctan nx}{\arctan n}$, $n = 1, 2, ...$ is the unique solution of the approximated problem.

(b) Show that $\int_{-1}^{+1} x^2\left(\frac{du_n}{dx}\right)^2 dx \to 0$ as $n \to \infty$.

(c) Conclude that there is no regular solution u of the original minimization problem.

(d) Why does the Lax-Milgram theorem fail in this case ?

[**4**] Let V be a Hilbert space and the (nonlinear) operator $A : V \to V$, satisfying

(i) there is $M \geq 0$ s.t. $\forall\, u, v \in V$, $\|Au - Av\| \leq M\|u - v\|$

(ii) there is $\alpha > 0$ s.t. $\forall\, u, v \in V$, $\langle Au - Av, u - v\rangle \geq \alpha\|u - v\|^2$

Show that the nonlinear equation $Au = f$ has a unique solution (for $f \in V$), using the Banach fixed point theorem and the same technique for proving the Lax-Milgram theorem (introduce the function g_λ).