Math 164/2 Comments:

- Reminder: Final Exam on Monday, March 20, time 3-6pm, room MS 5137.
- Sample problems for the final with solutions are posted on the class webpage. - Office Hours with the instructor before the final: Thursday 5pm, Friday 2-3pm (and at 5pm), Sunday 12-2pm (please note that I will not be able to hold regular office hour on Wednesday), and probably Monday before the final.
- Last homework due on Friday or on Monday.
- When you apply the Complementary Slackness Thm. 6.3, page 154, Chapter 6, make sure that the primal problem is in standard form.
- Problems 6 and 8 from the sample final problems are examples from Section 14.5.

Hints for the last homework #9:

Problem [2] I want you to use the general method (Lemmas 14.1 and 14.2), using a basis matrix Z of the null-space of A (and not the more "naive" method where we could express $x_3 = 3 - (2x_1 + 5x_2 + x_3)$ to obtain an unconstrained problem; this method works in this case too, but I want you to learn the general method introduced in Section 14.2).

Recall that x is a stationary point for problem [2] if $Ax = b$ and if $Z^T \nabla f(x) = \vec{0}$ (or equivalently, if $Ax = b$ and if $\nabla f(x) = A^T \lambda$ for some vector λ). Here $A = [2 \;\; 5 \;\; 1] \in \mathcal{M}(1 \times 3)$, $A^T \in \mathcal{M}(3 \times 1)$ and λ is a scalar (or vector with just one component).

Problem [3] You do not need to change the maximization problem into a minimization problem. You have to find local maximizers. Apply Lemma 14.2, page 431, with "negative definite" instead of "positive definite". First you need to find all stationary points, then see which ones are local maximizers, if any.

Here $A = [\frac{1}{a_1} \;\; \frac{1}{a_2} \;\; \frac{1}{a_3}] \in \mathcal{M}(1 \times 3)$ and $b = 1$. To find the stationary points first, find Z as a 3×2 matrix, then find all $x = (x_1, x_2, x_3)^T$ such that

$$\begin{cases} \frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} = 1 \\ Z^T \nabla f(x) = \vec{0}, \end{cases} \tag{1}$$

System (1) has 3 equations and 3 unknows x_1, x_2, x_3. You will show that all possible solutions of the system are $x = (\frac{a_1}{3}, \frac{a_2}{3}, \frac{a_3}{3})^T$, $x = (0, 0, a_3)^T$, $x = (0, a_2, 0)^T$ and $x = (a_1, 0, 0)^T$.

For each of the 4 points found, compute $Z^T \nabla^2 f(x) Z$ (a 2×2 matrix) and see if it is negative definite. Since the problem has some symmetry, it is sufficient to test for the first point and for one of the last 3 points only.

You will obtain that the first point $(\frac{a_1}{3}, \frac{a_2}{3}, \frac{a_3}{3})^T$ is a local maximizer (the 2×2 matrix $Z^T \nabla^2 f(x) Z$ can be transformed into an upper triangular matrix with all entries on the diagonal negative, so $Z^T \nabla^2 f(x) Z$ will be negative definite). The other 3 points are not local maximizers (the corresponding 2×2 matrices $Z^T \nabla^2 f(x) Z$ have one eigenvalue strictly positive and one eigenvalue strictly negative, therefore $Z^T \nabla^2 f(x) Z$ is not negative semi-definite, so the 2nd order necessary condition for local maximizer is not satisfied).

Problem [4] This can be solved as Example 14.8, page 443, from the textbook. Here again we do not know in advance the candidate points to local minimizers.

First write the constraints in the form $Ax \geq b$ (multiply the second constraint by -1):

$$\begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}.$$

There are 3 constraints, $A \in \mathcal{M}(3 \times 2)$, $A^T \in \mathcal{M}(2 \times 3)$, therefore the full Lagrange multiplier λ has 3 components $\lambda = (\lambda_1, \lambda_2, \lambda_3)^T$.

Necessary conditions for a point $x = (x_1, x_2)^T$ to be local minimizer are:

(1) $2x_1 + x_2 \geq 2$

(2) $-x_1 + x_2 \geq -1$

(3) $x_1 \geq 0$

together with $\nabla f(x) = A^T \lambda$ (two equations that you should write explicitely), $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $\lambda_3 \geq 0$, and complementary slackness $\lambda_1(2x_1 + x_2 - 2) = 0$, $\lambda_2(-x_1 + x_2 - (-1)) = 0$, $\lambda_3(x_1 - 0) = 0$.

To find the solutions of this set of conditions, consider the following cases, and in each case try to find the 5 variables $x_1, x_2, \lambda_1, \lambda_2, \lambda_3$, whenever a solution exists:

(i) All three constraints (1), (2), (3) are active (for example this should give you no solutions)

(iia) (1) and (2) active, $\lambda_3 = 0$ (here you should get $x_1 = 1$, $x_2 = 0$, $\lambda_1 = 1/3$, $\lambda_2 = -1/3$, $\lambda_3 = 0$, but this will not be a solution because $\lambda_2 < 0$).

(iib) (1) and (3) active, $\lambda_2 = 0$

(iic) (2) and (3) active, $\lambda_1 = 0$

(iiia) (1) active, $\lambda_2 = \lambda_3 = 0$

(iiib) (2) active, $\lambda_1 = \lambda_3 = 0$

(iiic) (3) active, $\lambda_1 = \lambda_2 = 0$

(iv) All three constraints are inactive, $\lambda_1 = \lambda_2 = \lambda_3 = 0$ (for example here you should get $(x_1, x_2) = (0, 0)$ which is not a feasible point, therefore no solution).

Finalize the other cases. For the solutions that you obtain, if any, make sure again that the strict complementary slackness holds. Also, for each such point, take $\hat{A}$ the submatrix of active constraints at that point, find its corresponding matrix Z and verify the second order conditions: $Z^T \nabla^2 f(x) Z$ positive definite, to obtain the local minimizer(s).