Summary: sufficient conditions for x_* strict local minimizer

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I. Min $f(x)$
• $\nabla f(x_*) = \vec{0}$
• $\nabla^2 f(x_*)$ positive definite

II. Min $f(x)$ subject to $Ax = b$ (linear equality constraints)
• $Ax_* = b$
• $Z^T \nabla f(x_*) = \vec{0}$ $\quad (\Leftrightarrow \nabla f(x_*) = A^T \lambda_*)$
• $Z^T \nabla^2 f(x_*) Z$ positive definite

(here Z is a basis matrix for $Null(A)$)

III. Min $f(x)$ subject to $Ax \geq b$ (linear inequality constraints)
•$Ax_* \geq b$
• $\nabla f(x_*) = A^T \lambda_*$
• $\lambda_* \geq \vec{0}$
• strict complementarity $\quad (\lambda_{*,i} = 0 \Leftrightarrow$ inequality i inactive constraint)
• $Z^T \nabla^2 f(x_*) Z$ positive definite

(Z is a basis matrix for $Null(\hat{A})$, $\hat{A}$ submatrix of active constraints at x_*)

IV. Min $f(x)$ subject to $g(x) = (g_1(x) \ ... \ g_m(x))^T = \vec{0}$ (nonlinear equality constraints)

Let $\mathcal{L}(x, \lambda) = f(x) - \sum_{i=1}^m \lambda_i g_i(x) = f(x) - \lambda^T g(x)$ and $Z(x_*)$ be a basis for the null space of $\nabla g(x_*)^T$ (we assume that the gradients of the constraints $\nabla g_i(x_*)$ are lin. ind.)
• $g(x_*) = \vec{0}$
• $\nabla_x \mathcal{L}(x_*, \lambda_*) = 0$ $\quad (\Leftrightarrow Z(x_*)^T \nabla f(x_*) = \vec{0})$
• $Z(x_*)^T \nabla^2_{xx} \mathcal{L}(x_*, \lambda_*) Z(x_*)$ positive definite

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V. Min $f(x)$ subject to $g(x) = (g_1(x) \ ... \ g_m(x))^T \geq \vec{0}$ (nonlinear inequality constraints)

Let $\mathcal{L}(x, \lambda)$ as above and $Z_+(x_*)$ be a basis for the null space of the submatrix of $\nabla g(x_*)^T$ corresponding to active constraints with positive Lagrange multipliers (we assume that the gradients $\nabla g_i(x_*)$ of the active constraints $g_i(x_*) = 0$ are lin. ind.)
• $g(x_*) \geq \vec{0}$
• $\lambda_* \geq \vec{0}$
• $\lambda_*^T g(x_*) = 0$ $\quad$ (complementary slackness)
• $Z_+(x_*)^T \nabla^2_{xx} \mathcal{L}(x_*, \lambda_*) Z_+(x_*)$ positive definite

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REMARKS:

• If $\nabla^2 f(x_*)$ positive definite, then $Z^T \nabla^2 f(x_*) Z$ is also positive definite (converse not true).

• If the matrix A (or $\hat{A}$) is non-singular (i.e. $Null(A)$ is the empty set), then Z does not exist, and the condition $Z^T \nabla^2 f(x_*) Z$ is trivially satisfied.

• Lemma 14.5 (textbook) is another sufficient condition for III, when only complementary slackness holds (not strict). Then use Z_+ basis matrix of $Null(\hat{A}_+)$, where $\hat{A}_+$ submatrix of A corresponding to active constraints and strictly positive Lagrange multipliers.

• In IV and V, $\nabla g(x_*)$ plays the role of the matrix A, when $g(x) = Ax - b$, and $\nabla^2_{xx} \mathcal{L}(x_*, \lambda_*)$ plays the role of $\nabla^2 f(x_*)$.