Exercise #6, page 446:

In the proof of necessary conditions for a local minimizer for non-linear minimization problems with linear inequality constraints (page 440) we needed the following linear algebra result:

Let A be an $m \times n$ matrix, with $m \leq n$ and of rank m (all rows of A are linearly independent). Let $e_1 = [1\ 0\ 0\ ...\ 0]^T \in R^m$ be the first vector of the canonical basis. Then there is a vector $p \in R^n$ such that $Ap = e_1$.

Proof. Let $v_1, ..., v_m$ be the m rows of the matrix A. Let

$$S = Span\{v_2, ..., v_m\}$$

be the linear subspace of R^n spanned by the linearly independent vectors $\{v_2, ..., v_m\}$. We know that R^n is an inner product space, and any vector $v \in R^n$ can be written as $v = p + q$, with $q \in S$ and $p \in S^{\perp}$ (the orthogonal subspace to S).

Therefore $v_1 = p + q$, with $q \in S$, $p \in S^{\perp}$. We have that $p \neq \vec{0}$ (indeed, if by contradiction $p = \vec{0}$, then $v_1 = q \in S$, impossible, because v_1 is not a linear combination of $v_2, ..., v_m$).

We also have $v_1 \cdot p = (p + q) \cdot p = p \cdot p + q \cdot p = p \cdot p$. Let $\hat{p} = \frac{p}{\|p\|^2}$, with $\|p\|^2 = p \cdot p$. Then $v_1 \cdot \hat{p} = v_1 \cdot \frac{p}{\|p\|^2} = 1$ and $v_2 \cdot \hat{p} = 0$, ..., $v_m \cdot \hat{p} = 0$.

This implies that $A\hat{p} = e_1$.