4. (a) Define what is mean by "the function $f$ defined on the convex set $S$ is a convex function". (b) Let $f$ and $g$ be convex functions on a convex set $S$. Define a function $h$ by

$$h(x) = rf(x) + sg(x)$$

for all $x$ in $S$, where $r$ and $s$ are positive real numbers. Prove that $h$ is also a convex function on $S$, making it clear in your proof how you are making use of the hypothesis that $r$ and $s$ are positive.

a) A function $f: S \to \mathbb{R}$, $S$ is a convex set is said to be a convex function if $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$, for every $\alpha \in [0,1]$ and for every $x, y \in S$.

b) Let $x, y \in S$ and $\alpha \in [0,1]$

$$h(\alpha x + (1-\alpha)y) = rf(\alpha x + (1-\alpha)y) + sg(\alpha x + (1-\alpha)y)$$

$$\overset{(*)}{\leq} r(\alpha f(x) + (1-\alpha) f(y)) + s(\alpha g(x) + (1-\alpha) g(y))$$

$$= \alpha (rf(x) + sg(x)) + (1-\alpha)(rf(y) + sg(y))$$

$$\cancel{= \alpha (f+g)(x) + (1-\alpha)(f+g)(y)}$$

$$= \alpha (rf + sg)(x) + (1-\alpha)(rf + sg)(y).$$

We used the fact that $r, s$ are positive at the inequality $(*)$.