4. (a) Define what is mean by the function \( f \) defined on the convex set \( S \) is a convex function. (b) Let \( f \) and \( g \) be convex functions on a convex set \( S \). Define a function \( h \) by
\[
h(x) = rf(x) + sg(x)
\]
for all \( x \in S \), where \( r \) and \( s \) are positive real numbers. Prove that \( h \) is also a convex function on \( S \), making it clear in your proof how you are making use of the hypothesis that \( r \) and \( s \) are positive.

a) A function \( f : S \to \mathbb{R} \), \( S \) is a convex set
is said to be a convex function if
\[
f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y), \quad \text{for every } \lambda \in [0,1] \text{ and for every } x, y \in S.
\]

b) Let \( x, y \in S \) and \( \lambda \in [0,1] \)
\[
\begin{align*}
h(\lambda x + (1-\lambda)y) &= \lambda f(\lambda x + (1-\lambda)y) + s g(\lambda x + (1-\lambda)y) \\
&\leq \lambda (\lambda f(x) + (1-\lambda)f(y)) + s(\lambda g(x) + (1-\lambda)g(y)) \\
&= \lambda (\lambda f(x) + s g(x)) + (1-\lambda)(\lambda f(y) + s g(y)) \\
&= \lambda f(\lambda x + (1-\lambda)y) + (1-\lambda)g(\lambda x + (1-\lambda)y). \quad \text{We used the fact that } \lambda, s \text{ are positive at the inequality } (\star).
\end{align*}
\]