The Simplex Method*

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Consider the linear programming problem

$$\begin{aligned} \text{minimize} \quad & z = -3x_1 - 4x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \le 6 \\ & x_1 + x_2 \le 4 \\ & x_1 - x_2 \le 2 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Refer to the figure handed out for sections 3.1 and 4.3 earlier this quarter.
First we need to convert the problem into standard form, yielding

$$\begin{aligned} \text{minimize} \quad & z = -3x_1 - 4x_2 \\ \text{subject to} \quad & x_1 + 2x_2 + x_3 = 6 \\ & x_1 + x_2 + x_4 = 4 \\ & x_1 - x_2 + x_5 = 2 \\ & x_1, x_2, x_3, x_4, x_5 \ge 0 \end{aligned}$$

Thus using the notation introduced in class we get

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} A_{3\times 2} & \underbrace{I_{3\times 3}}_{\text{non-singular}} \end{bmatrix}$$

A 'natural' choice for a first basis is $\{x_3, x_4, x_5\}$, as the corresponding columns in A, forming the matrix $I_{3\times 3}$, are clearly linearly independent. The corresponding basic feasible solution is $x = \begin{bmatrix} 0 & 0 & 6 & 4 & 2 \end{bmatrix}^T = x_a$ and the objective at x_a is $z = 0$.

Q: Is this an optimal solution? Is there a feasible descent direction?

To answer these questions, we express the

basic variable in terms of the non-basic variables,

*Please let me know if you find any errors or typos.

which is easy in this case (why?):

$$\begin{aligned} x_3 &= 6 - x_1 - 2x_2 && (1) \\ x_4 &= 4 - x_1 - x_2 && (2) \\ x_5 &= 2 - x_1 + x_2 && (3) \end{aligned}$$

We find

feasible directions,

by changing the value of <u>one</u> of the currently non-basic variables from zero to a positive value, i.e. by increasing it. What happens to $z = -3x_1 - 4x_2$? z decreases as x_1 or x_2 is increased. Thus $x = \begin{bmatrix} 0 & 0 & 6 & 4 & 2 \end{bmatrix}^T$

is not optimal.

We head in descent direction towards the next BFS (i.e. extreme point of S), i.e. to x_b or x_e by increasing x_1 or x_2, but not both (why?). We choose the 'steeper descent direction', i.e. we choose to increase x_2, because z decreases faster upon increasing x_2 (whether we arrive at an optimal solution indeed 'sooner' this way still does depend on α of course). So our feasible direction is $p = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$.

But

how far (step length $\alpha =?$)

can we go, i.e. by how much can we increase x_2 while still maintaining feasibility? As we keep $x_1 = 0$ non-basic the current basic variables change according to (compare to (1), (2) and (3)):

$$\begin{aligned} x_3 &= 6 - 2x_2 && (4) \\ x_4 &= 4 - x_2 && (5) \\ x_5 &= 2 + x_2 && (6) \end{aligned}$$

Given $p = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$

- as far as (4) is concerned, the maximal step length α we can go is $\alpha = 3$ (arriving at the BFS x_e),
- as far as (5) is concerned, the maximal step length α we can go is $\alpha = 4$ (arriving at BS, but not BFS x_g),
- as far as (6) is concerned, there is no limit on α (we move away from the third constraint).

What we did here, is a special case of the ratio test:

$$\alpha = \min_{1 \le i \le 3} \left\{ \frac{b_i}{a_{ij}} \ : \ a_{ij} \le 0 \right\} = \min\{3, 4\} = 3$$

Choosing $\alpha = 3$ yields $x_3 = 0$, i.e. x_3 leaves the basis and becomes non-basic variable, while $x_2 = 3$ enters our new (second) basis.

We arrived at the beginning of our second iteration with

$$x_B = \begin{bmatrix} x_2 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} \text{ and } x_N = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } x = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \\ 5 \end{bmatrix}$$

is our new BFS corresponding to $x_e = \begin{bmatrix} 0 & 3 \end{bmatrix}^T$ with objective $z = -12$. Before starting over again, we

> *express the objective as well as current basic variables in terms of the non-basic variables.*

Former was not needed in the first iteration, due to the choice of the first basis. Doing the above yields for the objective

$$z = -x_1 + 2x_3 - 12$$

and we see here that increasing x_3 from its current zero-value would increase z which is not desired, but increasing x_1 will 'improve' i.e. decrease z and hence we are not at an optimal solution yet.

etc ... etc ....