8. Show that $(0,-1)^T$ is a local minimizer for the problem

Minimize $f(x) = f(x_1, x_2) = 2x_1^2 + x_2$
subject to

active $\quad x_2 \geq x_1^2 - 1 \qquad -x_1^2 + x_2 + 1 \geq 0$

inactive $(\lambda_2 = 0) \quad x_1 \geq x_2 \qquad x_1 - x_2 \geq 0$

by demonstrating that $(0,-1)^T$ satisfies the sufficient conditions presented in the course.

$$\mathcal{L}(x,\lambda) = 2x_1^2 + x_2 - \lambda_1(-x_1^2 + x_2 + 1) - \overset{=0}{\lambda_2}(x_1 - x_2)$$

$$\nabla_x \mathcal{L}(x,\lambda) = \begin{bmatrix} 4x_1 + 2\lambda_1 x_1 \\ 1 - \lambda_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{so } \lambda_1 = 1 > 0$$

$$\nabla g_1(x) = \nabla(-x_1^2 + x_2 + 1) = \begin{bmatrix} -2x_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ at } (0,-1)$$

so $(0,-1)^T$ is a regular point and $Z_{(0,-1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\nabla_{xx}^2 \mathcal{L}(x,\lambda) = \begin{bmatrix} 4 + 2\lambda_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 0 \end{bmatrix}$$
(at $(0,-1)$)

$$Z(0,-1)^T \nabla_{xx}^2 \mathcal{L}(0,-1,1,0)\, Z(0,-1) = 6 > 0$$