7. Consider the problem

Minimize $f(x_1, x_2, x_3) = x_1^3 - x_1x_2 + x_3^2$
subject to

$$\begin{aligned} x_1 \quad - x_3 &= 1 \\ 2x_1 + x_2 \quad &= 2 \end{aligned}$$

(a) Find a basis null space matrix Z for the constraint matrix A of this problem.
(b) For $x = (1, -1, 1)^T$, find v and λ such that

$$\nabla f(x) = Zv + A^T\lambda$$

(a) $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad p_1 - p_3 = 0 \quad p_3 = p_1$ $\quad 2p_1 + p_2 = 0 \quad p_2 = -2p_1$ $\quad$ So $Z = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

(b) $\nabla f(x) = \begin{bmatrix} 3x_1^2 - x_2 \\ -x_1 \\ 2x_3 \end{bmatrix}$ $\quad \nabla f(1,-1,1) = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} v + \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$$\begin{cases} 4 = v + \lambda_1 + 2\lambda_2 \\ -1 = -2v + \lambda_2 \\ 2 = v - \lambda_1 \end{cases}$$

$v = \lambda_1 + 2$

$$\begin{cases} 4 = (\lambda_1 + 2) + \lambda_1 + 2\lambda_2 \\ -1 = -2(\lambda_1 + 2) + \lambda_2 \end{cases}$$

$$\begin{cases} 4 = 2\lambda_1 + 2\lambda_2 + 2 \\ -1 = -2\lambda_1 - 4 + \lambda_2 \end{cases}$$

$\lambda_1 + \lambda_2 = 1 \quad \lambda_1 = 1 - \lambda_2$

$3 = -2(1 - \lambda_2) + \lambda_2$

$3 = -2 + 3\lambda_2$

$\lambda_2 = 5/3 \quad \lambda_1 = 1 - 5/3 = -2/3 \quad v = -2/3 + 2 = 4/3$