

6. Show that $(-\sqrt{2/3}, \sqrt{1/6}, \sqrt{1/6})^T$ is a local minimizer for the problem

Minimize $f(x) = f(x_1, x_2, x_3) = 3x_1 + x_2 + x_3$
subject to

$$\begin{aligned}x_1 + x_2 + x_3 &= 0 \\x_1^2 + x_2^2 + x_3^2 &= 1\end{aligned}$$

by demonstrating that $(-\sqrt{2/3}, \sqrt{1/6}, \sqrt{1/6})^T$ satisfies the sufficient conditions presented in the course.

$$\text{feasible} : -\sqrt{\frac{2}{3}} + \sqrt{\frac{1}{6}} + \sqrt{\frac{1}{6}} = -\sqrt{\frac{2}{3}} + 2\sqrt{\frac{1}{6}} = 0$$

$$\frac{2}{3} + \frac{1}{6} + \frac{1}{6} = 1$$

$$\mathcal{L}(x, \lambda) = 3x_1 + x_2 + x_3 - \lambda_1(x_1 + x_2 + x_3) - \lambda_2(x_1^2 + x_2^2 + x_3^2 - 1)$$

$$\nabla_x \mathcal{L}(x, \lambda) = \begin{bmatrix} 3 - \lambda_1 + 2\lambda_2 x_1 \\ 1 - \lambda_1 + 2\lambda_2 x_2 \\ 1 - \lambda_1 + 2\lambda_2 x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(\text{add}) \quad 5 + 3\lambda_1 + 2\lambda_2(x_1 + x_2 + x_3) = 0 \quad \lambda_1 = \frac{5}{3}$$

$$3 - (\frac{5}{3}) - 2\lambda_2(-\sqrt{\frac{2}{3}}) = 0 \quad \frac{4}{3} = -2\sqrt{\frac{2}{3}}\lambda_2$$

$$\lambda_2 = -\sqrt{\frac{2}{3}}. \quad \text{Note } 1 - (\frac{5}{3}) - 2(-\sqrt{\frac{2}{3}})(\sqrt{\frac{1}{6}}) = 0$$

$$\nabla_{xx}^2 \mathcal{L}(x, \lambda) = \begin{bmatrix} -2\lambda_2 & 0 & 0 \\ 0 & -2\lambda_2 & 0 \\ 0 & 0 & -2\lambda_2 \end{bmatrix} = \begin{bmatrix} 2\sqrt{\frac{2}{3}} & 0 & 0 \\ 0 & 2\sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & 2\sqrt{\frac{2}{3}} \end{bmatrix}$$

positive definite so we know $\mathcal{Z}(x)^T \nabla_{xx}^2 \mathcal{L}(x, \lambda) \mathcal{Z}(x)$ is positive definite without calculating it.

$$\text{The rows of } \nabla g(-\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{6}}, \sqrt{\frac{1}{6}}) = \begin{bmatrix} 1 & 1 & 1 \\ -2\sqrt{\frac{2}{3}} & \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{6}} \end{bmatrix}$$

are linearly independent, so the point is regular.