10. Prove that if $x_*$ is a vector such that $\nabla f(x_*) = 0$ and $\nabla^2 f(x_*)$ is positive definite, then $x_*$ is a local minimizer for $f(x)$.

For $x$ near $x_*$, Taylor polynomial in remainder form:

$$f(x) = f(x_*) + p^T \nabla f(x_*) + \frac{1}{2} p^T \nabla^2 f(\xi) p$$

for $p = x - x_*$ and $\xi$ between $x$ and $x_*$.

Since $\nabla f(x_*) = 0$ then

$$f(x) = f(x_*) + \frac{1}{2} p^T \nabla^2 f(\xi) p$$

Since $\nabla^2 f(x_*)$ is positive definite, so also is $\nabla^2 f(\xi)$ and therefore

$$\frac{1}{2} p^T \nabla^2 f(\xi) p > 0$$

which implies

$$f(x) = f(x_*) + \frac{1}{2} p^T \nabla^2 f(\xi) p > f(x_*)$$

for all $x$ near $x_*$ and therefore $x_*$ is a local minimizer for $f(x)$. 