10. Prove that if x_* is a vector such that $\nabla f(x_*) = \mathbf{0}$ and $\nabla^2 f(x_*)$ is positive definite, then x_* is a local minimizer for $f(x)$.

For x near x_*, Taylor polynomial in remainder form

$$f(x) = f(x_*) + p^T \nabla f(x_*) + \frac{1}{2} p^T \nabla^2 f(\xi) p$$

for $p = x - x_*$ and ξ between x and x_*

Since $\nabla f(x_*) = \underline{0}$ then

$$f(x) = f(x_*) + \frac{1}{2} p^T \nabla^2 f(\xi) p$$

Since $\nabla^2 f(x_*)$ is positive definite, so also is $\nabla^2 f(\xi)$ and therefore $\frac{1}{2} p^T \nabla^2 f(\xi) p > 0$ which implies

$$f(x) = f(x_*) + \frac{1}{2} p^T \nabla^2 f(\xi) p > f(x_*)$$

for all x near x_* and therefore x_* is a local minimizer for $f(x)$.