1. Find all the local minimizers of the function

$$f(x_1, x_2) = (x_1 - \tfrac{1}{2}x_2)^2 + x_2^3 - 3x_2$$

and demonstrate that they satisfy the sufficient conditions presented in the class.

$$\nabla f(x_1, x_2) = \begin{bmatrix} 2(x_1 - \frac{1}{2}x_2) \\ 2(x_1 - \frac{1}{2}x_2)(-\frac{1}{2}) + 3x_2^2 - 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 = \frac{1}{2}x_2$ $3x_2^2 - 3 = 0$ $x_2 = \pm 1$ so $(\frac{1}{2}, 1)$ and $(-\frac{1}{2}, -1)$ are the stationary points.

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 2 & -1 \\ -1 & \frac{1}{2} + 6x_2 \end{bmatrix}$$

$$\nabla^2 f(\tfrac{1}{2}, 1) = \begin{bmatrix} 2 & -1 \\ -1 & 13/2 \end{bmatrix} \qquad \nabla^2 f(-\tfrac{1}{2}, -1) = \begin{bmatrix} 2 & -1 \\ -1 & -11/2 \end{bmatrix}$$

$\downarrow$ $\downarrow$

(pos def) $\begin{bmatrix} 2 & -1 \\ 0 & 6 \end{bmatrix}$ (not pos def) $\begin{bmatrix} 2 & -1 \\ 0 & -6 \end{bmatrix}$

$(\frac{1}{2}, 1)$ is the only local minimizer.