Summary: sufficient conditions for x_* strict local minimizer

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I. Min $f(x)$

$$\begin{cases} \nabla f(x_*) = \vec{0} \\ \nabla^2 f(x_*) \text{ positive definite} \end{cases}$$

II. Min $f(x)$ subject to $Ax = b$ (linear equality constraints, rows of A are linearly independent)

$$\begin{cases} Ax_* = b \\ Z^T \nabla f(x_*) = \vec{0} \Leftrightarrow \nabla f(x_*) = A^T \lambda_* \\ Z^T \nabla^2 f(x_*) Z \text{ positive definite} \end{cases}$$

(Z is a null space basis matrix of A).

III. Min $f(x)$ subject to $Ax \geq b$ (linear inequality constraints)

$$\begin{cases} Ax_* \geq b \\ \nabla f(x_*) = A^T \lambda_* \\ \lambda_* \geq \vec{0} \\ \text{strict complementarity } (\lambda_{*,i} = 0 \text{ iff inequality } i \text{ inactive constraint}) \\ Z^T \nabla^2 f(x_*) Z \text{ positive definite} \end{cases}$$

(Z is a null space basis matrix of $\hat{A}$, submatrix of active constraints at x_*, $\hat{A}$ has lin. ind. rows).

IV. Min $f(x)$ subject to $g(x) = (g_1(x) \;...\; g_m(x))^T = \vec{0}$ (nonlinear equality constraints)

Let $\mathcal{L}(x, \lambda) = f(x) - \sum_{i=1}^m \lambda_i g_i(x) = f(x) - \lambda^T g(x)$ and $Z(x_*)$ be a null space basis matrix of $\nabla g(x_*)^T$ (assume that gradients $\nabla g_i(x_*)$ are lin. ind.)

$$\begin{cases} g(x_*) = \vec{0} \\ \nabla_x \mathcal{L}(x_*, \lambda_*) = 0 \\ Z(x_*)^T \nabla^2_{xx} \mathcal{L}(x_*, \lambda_*) Z(x_*) \text{ positive definite} \end{cases}$$

V. Min $f(x)$ subject to $g(x) = (g_1(x) \;...\; g_m(x))^T \geq \vec{0}$ (nonlinear inequality constraints)

Let $\mathcal{L}(x, \lambda) = f(x) - \sum_{i=1}^m \lambda_i g_i(x) = f(x) - \lambda^T g(x)$ and $Z(x_*)$ be a null space basis matrix of the submatrix of $\nabla g(x_*)^T$ corresponding to active constraints (assume gradients $\nabla g_i(x_*)$ of active constraints $g_i(x_*) = 0$ are lin. ind.)

$$\begin{cases} g(x_*) \geq \vec{0} \\ \nabla_x \mathcal{L}(x_*, \lambda_*) = 0 \\ \lambda_* \geq \vec{0} \\ \text{strict complementarity } (\lambda_{*,i} = 0 \text{ iff inequality } i \text{ inactive constraint}) \\ Z(x_*)^T \nabla^2_{xx} \mathcal{L}(x_*, \lambda_*) Z(x_*) \text{ positive definite} \end{cases}$$

REMARKS:

- If $\nabla^2 f(x_*)$ positive definite, then $Z^T \nabla^2 f(x_*) Z$ is also positive definite (converse not true).
- If the matrix A (or $\hat{A}$) is non-singular (i.e. $Null(A)$ is the empty set), then Z does not exist, and the condition $Z^T \nabla^2 f(x_*) Z$ is trivially satisfied.
- In IV and V, $\nabla g(x_*)^T$ plays the role of the matrix A, when $g(x) = Ax - b$, and $\nabla^2_{xx} \mathcal{L}(x_*, \lambda_*)$ plays the role of $\nabla^2 f(x_*)$.
- Recall: $\nabla \vec{g}(x)^T$ has the gradients $\nabla g_i(x)$ written as rows.
- "Negative definite" instead of "positive definite" for local maximizer (and $\lambda \leq 0$ for inequality constraints).
- For III and V, the textbook gives other sufficiency conditions for the case of degenerate constraints (Lemma 14.5 and Thm. 14.4).