MATH 164/2 Optimization, Winter 2006, Midterm Exam Solutions
Instructor: Luminita Vese
Teaching Assistant: Ricardo Salazar

[**1**] (10 points)

(a) Consider the following feasible set $S = \left\{x \in R^n : Ax \geq b, \quad x \geq \vec{0}\right\}$. Show that if the vector d satisfies $d \neq \vec{0}$, $d \geq \vec{0}$ and $Ad \geq \vec{0}$, then d is a direction of unboundedness for the set S.

(b) Consider the following linear programming problem:

Minimize $z = x_1 - 5x_2 + x_3 - 3x_4$
subject to

$$\begin{array}{rcrcrcrcr} 3x_1 & - & x_2 & & & + & 2x_4 & \geq & 2 \\ -2x_1 & & & & & + & 3x_4 & \geq & -1 \\ & & x_2 & - & x_3 & & & \geq & 2 \\ 5x_1 & & & - & 3x_3 & & & \geq & 2 \\ & & x_1, & x_2, & x_3, & x_4 & & \geq & 0 \end{array}$$

(b1) Show that $x = (2,4,2,1)^T$ is a feasible point to the problem and label each of the constraints as active or inactive.

(b2) Show that the vector $d = (1,2,1,1)^T$ is a direction of unboundedness (note that this problem, as given, is not in standard form!)

Solution:

(a) We know by definition that d is a direction of unboundedness for S if $d \neq \vec{0}$ and if $x + \gamma d \in S$, for any $x \in S$ and any $\gamma \geq 0$.

Let $x \in S$ and $\gamma \geq 0$ arbitrary. Then $Ax \geq b$, and using the assumptions on d, we have:

$A(x + \gamma d) = Ax + A(\gamma d) = Ax + \gamma Ad \geq b + \gamma\vec{0} = b$.

Therefore $A(x + \gamma d) \geq b \quad (1)$

Also, because $x \geq \vec{0}$, $\gamma \geq 0$ and $d \geq \vec{0}$, we have $x + \gamma d \geq \vec{0} + \gamma\vec{0} = \vec{0} \quad (2)$

From (1) and (2), we deduce that $x + \gamma d \in S$ for any $x \in S$ and any $\gamma \geq 0$, therefore $d \neq 0$ is a direction of unboundedness for the set S.

(b1) We verify that $x = (2,4,2,1)^T$ satisfies all <u>eight</u> constraints given.

$3 \cdot 2 - 4 + 2 \cdot 1 = 4 > 2$ (1st inactive)

$-2 \cdot 2 + 3 \cdot 1 = -1 = -1$ (2nd active)

$4 - 2 = 2 = 2$ (3rd active)

$5 \cdot 2 - 3 \cdot 2 = 4 > 2$ (4th inactive)

$x_1 = 2 > 0$ (inactive), $x_2 = 4 > 0$ (inactive), $x_3 = 2 > 0$ (inactive), $x_4 = 1 > 0$ (inactive).

Therefore the point $x = (2,4,2,1)^T$ satisfies all eight constraints of the problem, so it is a feasible point, $x \in S$.

(b2) Notice that $d = (1,2,1,1)^T \neq \vec{0}$ and $d \geq \vec{0}$. Therefore, using the proof for (a), it is sufficient to show that $Ad \geq \vec{0}$, where A is the matrix corresponding to the first four constraints:

$$A = \begin{pmatrix} 3 & -1 & 0 & 2 \\ -2 & 0 & 0 & 3 \\ 0 & 1 & -1 & 0 \\ 5 & 0 & -3 & 0 \end{pmatrix}.$$

Indeed,

$$Ad = \begin{pmatrix} 3 & -1 & 0 & 2 \\ -2 & 0 & 0 & 3 \\ 0 & 1 & -1 & 0 \\ 5 & 0 & -3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 2 \end{pmatrix} > \vec{0} \geq \vec{0},$$

therefore $d = (1, 2, 1, 1)^T$ is a direction of unboundedness for S given in (b).

[**2**] (12 points) Consider the linear programming problem

Minimize $z = x_1 - x_2 + 3x_3$
subject to

$$\begin{array}{rcrcrcr} x_1 & & & + & 2x_3 & \geq & 4 \\ x_1 & - & x_2 & & & \geq & 0 \\ -2x_1 & + & x_2 & + & x_3 & \geq & 1 \\ & & x_1, & x_2, & x_3 & \geq 0 & \end{array}$$

(a) Show that $x = (1, 1, 2)^T$ is a feasible point to the problem.
(b) Show that $p = (-2, -3, 0)^T$ is a feasible direction at the feasible point $x = (1, 1, 2)^T$.
(c) Determine the maximal step length α such that $x + \alpha p$ remains feasible, with x and p as in part (b).
(d) Find the minimum value of p_3, such that $(-2, -3, p_3)^T$ is a feasible direction at $x = (1, 1, 2)^T$.

Solution:
We will also label all six constraints, since this is needed for (b)-(d).
(a)
$1 + 2 \cdot 2 = 5 > 4$ (inactive)
$1 - 1 = 0 = 0$ (active)
$(-2) \cdot 1 + 1 + 2 = 1 = 1$ (active)
$x_1 = 1 > 0$ (inactive), $x_2 = 1 > 0$ (inactive), $x_3 = 2 > 0$ (inactive).
Therefore $x = (1, 1, 2)^T$ is a feasible point.

(b) By the property from the course, it is sufficient to show that $\hat{A}p \geq \vec{0}$, where $\hat{A}$ is the submatrix of A corresponding to active constraints only:

$$\hat{A}p = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \geq \vec{0},$$

therefore $p = (-2, -3, 0)^T$ is a feasible direction at $x = (1, 1, 2)^T$.

(c) We know that only the inactive constraints determine the max step length alpha.

$x + \alpha p = (1-2\alpha, 1-3\alpha, 2)^T$, then α_{max} is obtained from imposing that this point $x + \alpha p$ satisfies all inactive constraints, as follows:

$[1\ 0\ 2]\begin{bmatrix} 1-2\alpha \\ 1-3\alpha \\ 2 \end{bmatrix} = 1 - 2\alpha + 4 \geq 4 \Rightarrow \alpha \leq \frac{1}{2}$

$1 - 2\alpha \geq 0 \Rightarrow \alpha \leq \frac{1}{2}$

$1 - 3\alpha \geq 0 \Rightarrow \alpha \leq \frac{1}{3}$

$2 \geq 0 \Rightarrow$ no restriction on α.

The intersection of all 4 above conditions an α will give us $\alpha \leq \alpha_{max} = \frac{1}{3}$.

Note that the ratio test could have been used, and it would provide the same answer (exercise).

(d) We proceed as in (b). We need to impose that $\hat{A}p \geq \vec{0}$, or that

$$\hat{A}p = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ p_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 + p_3 \end{bmatrix} \geq \vec{0},$$

therefore we need $1 + p_3 \geq 0$, or $p_3 \geq -1$.

[3] In solving a linear (minimization) programming problem by the simplex method, we arrive at the objective function in the form

$$z = 4x_3 - 2x_4 + 3x_5 + 2,$$

and the dictionary

$$x_1 = x_3 - 3x_4 - 3x_5 + 4$$

$$x_2 = 2x_3 - x_4 + x_5 + 1.$$

Use the simplex algorithm to find the optimal solution to the minimization problem.

Solution:

At this step, $x_B = \{x_1, x_2\}$ and $x_N = \{x_3, x_4, x_5\} = \{0, 0, 0\}$. The corresponding basic feasible solution is $(4, 1, 0, 0, 0)^T$, with $z(4, 1, 0, 0, 0) = 2$. We notice that x_4, now zero, has the negative coefficient -2 inside z, therefore z decreases further if x_4 is increased from 0 to a positive value. Therefore x_4 enters the basis, and we keep $x_3 = x_5 = 0$ outside the basis.

At this new basic feasible solution with $x_3 = x_5 = 0$, we need to have:

$x_1 = -3x_4 + 4 \geq 0 \Rightarrow x_4 \leq \frac{4}{3}$ and

$x_2 = -x_4 + 1 \geq 0 \Rightarrow x_4 \leq 1$.

The intersection gives us $x_4 = 1$, therefore $x_2 = 0$ leaves the basis. The new basis is $x_B = \{x_1, x_4\}$, $x_N = \{x_2, x_3, x_5\} = \{0, 0, 0\}$, with the new basic feasible solution $(1, 0, 0, 1, 0)^T$.

The new dictionary is:

$x_4 = -x_2 + 2x_3 + x_5 + 1$,

$x_1 = x_3 - 3(-x_2 + 2x_3 + x_5 + 1) - 3x_5 + 4 = 3x_2 - 5x_3 - 6x_5 + 1$, or the new dictionary is

$x_1 = 3x_2 - 5x_3 - 6x_5 + 1$,
$x_4 = -x_2 + 2x_3 + x_5 + 1$ and the new z function of non-basic variables is
$z = 4x_3 - 2x_4 + 3x_5 + 2 = 4x_3 - 2(-x_2 + 2x_3 + x_5 + 1) + 3x_5 + 2 = +2x_2 + x_5$.

Notice now all variables inside z have postive or zero coefficients, therefore the basic feasible solution $(1,0,0,1,0)^T$ is optimal, with optimal value

$$min(z) = z(1,0,0,1,0) = 0.$$

[4] (7 points) Suppose that a linear program has l optimal extreme points $\{v_1, v_2, ..., v_l\}$. Prove that if a feasible point x can be expressed as a convex combination of v_i, then x is optimal.

Solution: Let x be a convex combination of $v_1, v_2, ..., v_l$, with $x = \sum_{i=1}^{l} \alpha_i v_i$, for some $\alpha_i \geq 0$ and $\sum_{i=1}^{l} \alpha_i = 1$.

Let $M := min_S z$, with $z(x) = c^T x$. Then $z(v_i) = c^T v_i = M$, for all $i = 1, 2, ..., l$.

We have: $z(x) = c^T x = c^T \Big(\sum_{i=1}^{l} \alpha_i v_i \Big) = \sum_{i=1}^{l} c^T (\alpha_i v_i) = \sum_{i=1}^{l} \alpha_i (c^T v_i) = \sum_{i=1}^{l} \alpha_i M = M \sum_{i=1}^{l} \alpha_i = M \cdot 1 = M = Min_{y \in S} z(y)$, therefore x is also an optimal solution of the linear programming problem.

(we use the fact that the function z is linear, the linearity was shown in class).

[5] (11 points)

(a) Recall the definitions of a convex set S and of a convex function g on S.

(b) Let g be a convex function on R^n. Prove that the set $S = \Big\{ x : \ g(x) \leq 0 \Big\}$ is convex.

(c) Let g_1, g_2 be two convex functions on the real line, and let $r > 0$ be a fixed real number. Show that the function $f(x) = x + g_1(x) + r g_2(x)$ is also convex on the real line.

Solution:

(a) The set S is convex if for any $x, y \in S$ and any $0 \leq \alpha \leq 1$, we have $\alpha x + (1-\alpha) y \in S$.

The function $g : S \to R$ is convex on the convex set S if for any $x, y \in S$ and any $0 \leq \alpha \leq 1$, we have $g(\alpha x + (1-\alpha) y) \leq \alpha g(x) + (1-\alpha) g(y)$.

(b) Let $x, y \in S$ and let $0 \leq \alpha \leq 1$ be arbitrary. Then $g(x) \leq 0$ and $g(y) \leq 0$. We also have, by convexity of g:

$g(\alpha x + (1-\alpha) y) \leq \alpha g(x) + (1-\alpha) g(y) \leq \alpha \cdot 0 + (1-\alpha) \cdot 0 = 0$ (because $\alpha \geq 0$ and $1 - \alpha \geq 0$), therefore $g(\alpha x + (1-\alpha) y) \leq 0$, or $\alpha x + (1-\alpha) y \in S$. In conclusion, the set S is convex.

(c) Let $x, y \in R$ and $0 \leq \alpha \leq 1$ be arbitrary. We have
$f(\alpha x + (1-\alpha) y) = \alpha x + (1-\alpha) y + g_1(\alpha x + (1-\alpha) y) + r g_2(\alpha x + (1-\alpha) y) \leq$
by convexity of g_1 and g_2 and $r > 0$
$\leq \alpha x + (1-\alpha) y + \alpha g_1(x) + (1-\alpha) g_1(y) + r(\alpha g_2(x) + (1-\alpha) g_2(y))$
$= \alpha x + \alpha g_1(x) + \alpha r g_2(x) + (1-\alpha) y + (1-\alpha) g_1(y) + (1-\alpha) r g_2(y)$
$= \alpha(x + g_1(x) + r g_2(x)) + (1-\alpha)(y + g_1(y) + r g_2(y)) = \alpha f(x) + (1-\alpha) f(y)$.

In conclusion, f is a convex function.