Math 164: Homework #5, due on Friday, February 13

[1] Suppose that a linear program in standard form, with bounded feasible region, has \( l \) optimal extreme points \( \{ v_1, v_2, ..., v_l \} \). Prove that a point is optimal for the linear program if, and only if, it can be expressed as a convex combination of \( \{ v_1, v_2, ..., v_l \} \).

[2] Solve the following linear program using the simplex method (graph the feasible region, and outline the progress of the algorithm).

\[
\text{maximize } z = 7x_1 + 8x_2, \quad \text{subject to } \begin{cases} 4x_1 + x_2 \leq 100 \\ x_1 + x_2 \leq 80 \\ x_1 \leq 40 \\ x_1, x_2 \geq 0. \end{cases}
\]

[3] Compute a basis matrix for the null space of the matrix \( A \) and express the point \( x \) as \( x = p + q \), where \( p \) is in the null space of \( A \) and \( q \) is in the range space of \( A^T \):

\[
A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad x = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 2 \end{pmatrix}.
\]

[4] Consider the linear program: Minimize \( z = x_1 - x_2 \) subject to

\[
\begin{align*}
-x_1 + x_2 & \leq 1 \\
x_1 - 2x_2 & \leq 2 \\
x_1, x_2 & \geq 0.
\end{align*}
\]

Derive an expression for the set of optimal solutions to this problem, and show that this set is unbounded.

[5] Find the dual of

\[
\text{minimize } z = 3x_1 - 5x_2 - 7x_3, \\
\text{subject to } \begin{cases} x_1 + 5x_2 - 8x_3 \geq 5 \\ 4x_1 - 2x_2 + 7x_3 \geq 7 \\ x_1, x_2, x_3 \geq 0. \end{cases}
\]