Math 164: Homework #4, due on Friday, February 6

• Review Sections 4.1-4.4 from the textbook. Please solve additional problems.

[**1**] Convert the following linear program to standard form:

$$\text{minimize } z = x_1 - 5x_2 - 7x_3, \text{ subject to } \begin{cases} 5x_1 - 2x_2 + 6x_3 \geq 5 \\ 3x_1 + 4x_2 - 9x_3 = 3 \\ 7x_1 + 3x_2 + 5x_3 \leq 9 \\ x_1 \geq -2, \ x_2, \ x_3 \text{ free.} \end{cases}$$

[**2**] Consider the system of linear constraints

$$\begin{array}{ccccc} 2x_1 & + & x_2 & \leq & 100, \\ x_1 & + & x_2 & \leq & 80, \\ x_1 & & & \leq & 40, \\ x_1 & , & x_2 & \geq & 0. \end{array}$$

(a) Write this system in standard form, and determine all the basic solutions (feasible and infeasible).

(b) Determine the extreme points of the feasible region (corresponding to both the standard form, as well as the original version).

[**3**] Consider a linear program with the constraints in standard form:

$$\{Ax = b, \ x \geq 0\}.$$

Prove that if $d \neq 0$ satisfies $Ad = 0$ and $d \geq 0$, then d is a direction of unboundedness.

[**4**] Consider the linear program: minimize $z = -5x_1 - 7x_2$, subject to $\begin{cases} -3x_1 + 2x_2 \leq 30 \\ -2x_1 + x_2 \leq 12 \\ x_1, x_2 \geq 0. \end{cases}$

(a) Draw a graph of the feasible region

(b) Determine the extreme points of the feasible region

(c) Determine two linearly-independent directions of unboundedness.

(d) Represent the point $x = (6, 12)^T$ as a convex combination of extreme points, plus if applicable, a direction of unboundedness.

(e) Show by the method of your choice that this problem has no finite optimal solution.

(f) Convert the linear program to standard form and determine two linearly-independent directions of unboundedness for this version of the problem. Verify that the directions of unboundedness satisfy $Ad = \vec{0}$ and $d \geq \vec{0}$.

[**5**] Consider a linear program with the constraints in standard form

$$Ax = b \text{ and } x \geq \vec{0}.$$

(a) Prove that, if d is a direction of unboundedness for these constraints, then $-d$ cannot be a direction of unboundedness.

(b) Let $\{d_1, ..., d_k\}$ be directions of unboundedness for these constraints. Prove that a nonzero vector $d = \sum_{i=1}^{k} \alpha_i d_i$, with $\alpha_i \geq 0$ is also a direction of unboundedness.