Math 164: Homework #3, due on Friday, January 30

Please note: exceptionally there will be no office hours with the instructor the week of Jan. 26. Please email if you have questions or see the T.A.

[1] Prove that a function \( f \) is concave if and only if \( -f \) is convex.

[2] Let \( f \) be a convex function on the convex set \( S \) of \( \mathbb{R}^n \). Let \( k \) be a nonzero scalar, and define \( g(x) = kf(x) \). Prove that if \( k > 0 \) then \( g \) is a convex function on \( S \), and if \( k < 0 \) then \( g \) is a concave function of \( S \).

[3] Consider a feasible region \( S \) defined by a set of linear constraints

\[
S = \{ x : Ax \leq b \},
\]

where \( A \) is an \( m \times n \) matrix and \( b \) is a column vector.

(a) Prove that \( S \) is convex.

(b) Derive the conditions that must be satisfied by a feasible direction \( p \) at a point \( x \in S \).

[4] Consider the problem

\[
\text{minimize } f(x)
\]

subject to \( x_1 + 2x_2 + 3x_3 = 6 \), \( x_1 \geq 0 \), \( x_2 \geq 0 \), \( x_3 \geq 0 \).

(a) Find the sets of all feasible directions at points \( x_a = (0, 0, 2)^T \), \( x_b = (3, 0, 1)^T \), and \( x_c = (1, 1, 1)^T \).

(b) Using (a), verify that \( p = (3, 0, -1)^T \) is a feasible direction for \( x_c = (1, 1, 1)^T \); then find an upper bound on the step length \( \alpha \) so that \( x_c + \alpha p \) is a feasible point, with \( p = (3, 0, -1)^T \).

[5] Solve the following linear program graphically:

\[
\text{maximize } z = 6x_1 - 3x_2, \text{ subject to } \begin{cases} 2x_1 + 5x_2 \geq 10, \\ 3x_1 + 2x_2 \leq 40, \\ x_1, x_2 \leq 15. \end{cases}
\]