In the proof of necessary conditions for a local minimizer for non-linear minimization problems with linear inequality constraints (page 440) we needed the following linear algebra result:

Let $A$ be an $m \times n$ matrix, with $m \leq n$ and of rank $m$ (all rows of $A$ are linearly independent). Let $e_1 = [1 \ 0 \ 0 \ ... \ 0]^T \in \mathbb{R}^m$ be the first vector of the canonical basis. Then there is a vector $p \in \mathbb{R}^n$ such that $Ap = e_1$.

**Proof.** Let $v_1, ..., v_m$ be the $m$ rows of the matrix $A$. Let

$$S = \text{Span}\{v_2, ..., v_m\}$$

be the linear subspace of $\mathbb{R}^n$ spanned by the linearly independent vectors $\{v_2, ..., v_m\}$. We know that $\mathbb{R}^n$ is an inner product space, and any vector $v \in \mathbb{R}^n$ can be written as $v = p + q$, with $q \in S$ and $p \in S^\perp$ (the orthogonal subspace to $S$).

Therefore $v_1 = p + q$, with $q \in S$, $p \in S^\perp$. We have that $p \neq \vec{0}$ (indeed, if by contradiction $p = \vec{0}$, then $v_1 = q \in S$, impossible, because $v_1$ is not a linear combination of $v_2, ..., v_m$).

We also have $v_1 \cdot p = (p + q) \cdot p = p \cdot p + q \cdot p = p \cdot p$. Let $\hat{p} = \frac{p}{\|p\|^2}$, with $\|p\|^2 = p \cdot p$. Then $v_1 \cdot \hat{p} = v_1 \cdot \frac{p}{\|p\|^2} = 1$ and $v_2 \cdot \hat{p} = 0$, ..., $v_m \cdot \hat{p} = 0$.

This implies that $A\hat{p} = e_1$. 

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