Problems and Computer Projects #4

[1] (a) Implement the total variation minimization model for image denoising (use as input an image degraded by additive Gaussian noise).

(b) Implement the total variation minimization model for image deblurring-denoising (with a very small amount of Gaussian noise).

[2] (a) Show that the magnitude of the gradient $|\nabla f| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$ is an isotropic operation. You will need the following equations relating coordinates after axis rotation by an angle θ :

$$x = x' \cos \theta - y' \sin \theta$$
$$y = x' \sin \theta + y' \cos \theta$$

where (x, y) are the unrotated and (x', y') are the rotated coordinates.

(b) Show that the isotropic property is lost in general if the gradient magnitude is approximated by $|\nabla f| \approx |f_x| + |f_y|$.

[3] (a) Show using Taylor's expansion that the finite differences formula $\frac{f(x+h)-f(x-h)}{2h}$ is a second order approximation of the first-order derivative f'(x).

(b) Show using Taylor's expansion that the finite differences formula $\frac{f(x+h)-f(x)}{h}$ is a first order approximation of the first-order derivative f'(x).

(c) Apply such formula to approximate the gradient map square

$$g(x,y) = |\nabla f|^2(x,y) = \left(\frac{\partial f(x,y)}{\partial x}\right)^2 + \left(\frac{\partial f(x,y)}{\partial y}\right)^2.$$

(d) Download Fig5.26a and plot an image of its gradient map g (rescaling may be necessary). Explain the steps taken. Ignore the pixels on the boundary of the image for simplicity, when computing the discrete gradient.

[4] (a) Recall the definition of the convolution f * g(x, y) in continuous variables and in two dimensions.

(b) Show that $\triangle(f * g) = f * (\triangle g) = (\triangle f) * g$, where \triangle denotes the Laplacian operator in (x, y).

[5] (a) Give the main steps of edge finding using the zero-crossings of the Laplacian of Gaussian.

(c) Implement and apply this method to the angiogram image from Fig. 10.15(a).

[6] Recall the 1D Laplacian of Gaussian (LoG) operator

$$\nabla^2 h(r) = \left[\frac{r^2 - \sigma^2}{\sigma^4}\right] e^{-\frac{r^2}{2\sigma^2}}.$$

(a) Show that the average value of the LoG operator $\nabla^2 h$ is zero. Hint: use the identities $\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{+\infty} e^{-r^2/(2\sigma^2)}dr = 1, \ \sigma^2 = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{+\infty} r^2 e^{-r^2/(2\sigma^2)}dr.$

(b) Prove that the average value of any function convolved with this operator is also zero. Hint: use the frequency domain.