Problems #2

[1] Recall that the one-dimensional Discrete Fourier Transform is

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-2\pi i u x / M}, \tag{1}$$

for all $u = 0, 1, 2, ..., M-1$.

(a) Assuming the formula above, show the identity

$$f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{2\pi i u x / M}, \tag{2}$$

using the following orthogonality of exponentials

$$\sum_{u=0}^{M-1} e^{-2\pi i u y / M} e^{2\pi i u x / M} = \begin{cases} M & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

(b) Show now the converse of (a): assume given $f(x)$ by (2) function of $F(u)$, and prove the identity (1) for $F(u)$ (use the same orthogonality of exponentials).

[2] Recall that the 2D Fourier transform in the continuous case is

$$F(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-2\pi i (ux+vy)} dx dy.$$

Show that the tranformation $\mathcal{F} : f \mapsto F$ is a linear transformation.

[3] Compute in continuous variables the Fourier transform of the function

$$f(x) = \begin{cases} A, & \text{if } 0 \le x \le K, \\ 0, & \text{otherwise,} \end{cases}$$

where A and K are positive constants. Evaluate $F(0)$.

(recall that $F(u) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i u x} dx$).

[4] Consider again the 2D continuous Fourier transform defined above, and its inverse defined by

$$f(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) e^{2\pi i (ux+vy)} du dv.$$

Show that if the transform $F(u,v)$ is real and symmetric, i.e. if

$$F(u,v) = \overline{F(u,v)} = \overline{F(-u,-v)} = F(-u,-v),$$

then the corresponding spatial domain function $f(x,y)$ is also real and symmetric.

We first recall here the discrete Fourier transforms and their inverses in one and two dimensions.

$$\text{1D DFT:} \quad F(u) = \sum_{x=0}^{M-1} f(x) e^{-2\pi i \frac{ux}{M}}, \quad \text{for } u = 0, 1, 2, ..., M-1.$$

$$\text{1D IDFT:} \quad f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{2\pi i \frac{ux}{M}}, \quad \text{for } x = 0, 1, 2, ..., M-1.$$

$$\text{2D DFT:} \quad F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-2\pi i (\frac{ux}{M} + \frac{vy}{N})}, \quad \text{for } u = 0, 1, 2, ..., M-1, \ \ v = 0, 1, 2, ..., N-1.$$

$$\text{2D IDFT:} \quad f(x,y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{2\pi i (\frac{ux}{M} + \frac{vy}{N})}, \quad \text{for } x = 0, 1, 2, ..., M-1, \ \ y = 0, 1, 2, ..., N-1.$$

[5] (a) Show in discrete variables that

$$\mathcal{F}\Big(f(x,y) e^{2\pi i (u_0 \frac{x}{M} + v_0 \frac{y}{N})}\Big) = F(u - u_0, v - v_0),$$

where $F = \mathcal{F}(f)$.

(b) Using (a), deduce the formula used in shifting the center of the transform by multiplication with $(-1)^{x+y}$, when $u_0 = M/2$ and $v_0 = N/2$, with M and N even positive integers.

[6] Show the translation property

$$\mathcal{F}\Big(f(x - x_0, y - y_0)\Big) = F(u,v) e^{-2\pi i (x_0 u/M + y_0 v/N)},$$

where $F(u,v) = \mathcal{F}(f(x,y))$.

[7] Consider the linear difference operator $g(x,y) = f(x+1,y) - f(x,y)$. Obtain the filter transfer function, $H(u,v)$, for performing the equivalent process in the frequency domain.

[8] Prove the validity of the discrete convolution theorem in one variable (you may need to use the translation properties).

[9] Assume that $f(x)$ is given by the discrete IFT formula in one dimension. Show the periodicity property $f(x) = f(x + kM)$, where k is an integer.

[10] Consider the two-dimensional continuous case.

(a) Recall the Inverse Fourier transform formula in 2D (express $f(x,y)$ function of $F(u,v)$).

(b) Assume f is twice differentiable. Using (a), find the Fourier transform of the mixed partial derivative $\frac{\partial^2 f}{\partial x \partial y}(x,y)$, function of F.