Problems #1

[**1**] (Histogram equalization in continuous variables) An image has the gray-level PDF

$$p_r(r) = \begin{cases} \frac{6r+2}{3(L-1)^2+2(L-1)} & \text{if } 0 \le r \le L-1 \\ 0, & \text{otherwise} \end{cases}$$

with $L-1>0$.

(a) Verify some of the properties that a PDF has to satisfy: $p_r(r) \ge 0$ for all $r \in (-\infty,\infty)$ and $\int_{-\infty}^{\infty} p_r(r)dr = 1$.

(b) Find the tranformation function $s = T(r)$ obtained through "histogram equalization" in continuous variables.

(c) Verify that $p_s(s)$ is a uniform "flat" distribution for $s \in [0,1]$ (recall the formula $p_s(s) = p_r(r)\left|\frac{dr}{ds}\right|$).

[**2**] A linear spatial filter of size $(2a+1)\times(2b+1)$ defined by the transformation H, $g = H[f]$, is given by

$$g(x,y) = \sum_{s=-a}^{a}\sum_{t=-b}^{b} w(s,t)f(x+s,y+t),$$

where $f(x,y)$ is a given input image, a,b are positive integers, and $w(s,t)$ are weights for $-a \le s \le a$, $-b \le t \le b$.

(a) Give the definition of a linear transformation $H : V \to V$, where V is a vector space.

(b) Show that H defined above is indeed a linear transformation (assume images defined on the entire plane, or ignore border effects).

[**3**]Show that the continuous Laplacian is a linear operation, in other words show that the mapping $f \mapsto \nabla^2 f$ is linear on the vector space of functions $f \in C^2$ in two dimensions.

[**4**] Show that the Laplacian operation $\triangle f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ is isotropic (invariant under rotations, or rotationally invariant). You will need the following equations relating coordinates after axis rotation by an angle θ:

$$x = x'\cos\theta - y'\sin\theta$$

$$y = x'\sin\theta + y'\cos\theta$$

where (x,y) are the unrotated and (x',y') are the rotated coordinates.

[**5**] (a) Show using Taylor's expansion that the finite differences formula $\frac{f(x+h)-2f(x)+f(x-h)}{h^2}$ is a second order approximation of the 2nd-order derivative $f''(x)$.

(b) Derive the 5-point Laplacian approximation using the result in (a).