UCLA MATH 151A/2, WINTER 2007, MIDTERM EXAM

NAME ______________________________ **STUDENT ID #** ____________________

This is a closed-book and closed-note examination. No calculators are allowed. Please show all your work. Partial credit will be given to partial answers. There are 5 problems of total 100 points.

You do not have to completely carry out the algebraic calculations.

PROBLEM	1	2	3	4	5	TOTAL
SCORE						

I. Let $f(x) = 3x - e^x$, and the table

x	1	1.125	1.250	1.375	1.500	1.625	1.750	1.875	2
$f(x)$	0.2817	0.2948	0.2597	0.1699	0.0183	-0.2034	-0.5046	-0.8958	-1.3891

(a) Prove that the equation $f(x) = 0$ has at least a solution p in the interval $[1, 2]$.

Solution: From the table, we see that $f(1) = 0.2817 > 0$, while $f(2) = -1.3891$. Also, f is continuous on $[1, 2]$, thus by the Intermediate Value Thm., there must be a $p \in (1, 2)$ such that $f(p) = 0$.

(b) By the Bisection method, find p_n, $n \leq 2$ on $[1, 2]$, and write your answers in the next table.

n	a_n	b_n	p_n	$f(p_n)$
0	1	2	1.5	0.0183
1	1.5	2	1.75	-0.5046
2	1.5	1.75	1.625	-0.2034

Solution:

(c) How many iterations are necessary to solve $3x - e^x = 0$ with accuracy 10^{-4} on $[1, 2]$?

Solution: From the theorem from the course, we impose

$$|p_n - p| \leq \frac{b-a}{2^n} \leq 10^{-4},$$

thus we impose $\frac{2-1}{2^n} \leq 10^{-4}$, or $10^4 \leq 2^n$.

This will give $\log_{10} 10^4 \leq \log_{10} 2^n$, or $4 \leq n \log_{10} 2$. Finally, n must be the smallest integer larger or equal to $\frac{4}{\log_{10} 2}$.

II.

(a) Use the Theorem from the course to prove that $g(x) = 2^{-x}$ has a unique fixed point on $[\frac{1}{3}, 1]$.

Solution: Note that g is continuous and differentiable in $[\frac{1}{3}, 1]$, with $g'(x) = -2^{-x} \ln 2 < 0$ on $[\frac{1}{3}, 1]$.

Thus g is monotonically decreasing on $[\frac{1}{3}, 1]$, and $g(1/3) \geq g(x) \geq g(1)$, or $1 > \frac{1}{2^{1/3}} \geq g(x) \geq \frac{1}{2} > \frac{1}{3}$ for any $x \in [\frac{1}{3}, 1]$. We deduce that

$$g(x) \in [\frac{1}{3}, 1], \text{ for any } x \in [\frac{1}{3}, 1].$$

We have $|g'(x)| = 2^{-x} \ln 2 \leq 2^{-1/3} \ln 2 = \frac{1}{2^{1/3}} \ln 2 = k < 1$.
Therefore, by the Fixed Point Theorem, there is a point $p \in [\frac{1}{3}, 1]$ such that $g(p) = p$.

(b) For $p_0 = 1/2$, compute p_1.

Solution: The fixed point iteration is $p_{n+1} = g(p_n)$, thus $p_1 = g(p_0) = 2^{-1/2} = \frac{1}{\sqrt{2}}$.

III.
(a) Recall Newton's method for solving $f(p) = 0$, with $f'(p) \neq 0$.

Solution: For given p_0, with $f'(p_0) \neq 0$, the Newton's iteration is

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}, \quad n \geq 0.$$

(b) Derive the Secant method from Newton's method, using an approximation to the derivative.

Solution: We use finite differences to approximate

$$f'(p_n) \approx \frac{f(p_n) - f(p_{n-1})}{p_n - p_{n-1}}.$$

Thus the Secant method is obtained by substituting $f'(p_n)$ by this approximation in Newton's method:

$$p_{n+1} = p_n - \frac{f(p_n)}{\frac{f(p_n)-f(p_{n-1})}{p_n-p_{n-1}}}, \quad n \geq 1,$$

knowing p_0 and p_1. Finally, the Secant Method is:
Given p_0 and p_1, compute for $n \geq 1$ the sequence

$$p_{n+1} = p_n - \frac{f(p_n)(p_n - p_{n-1})}{f(p_n) - f(p_{n-1})}.$$

IV. Let $f(x) = \ln(x + 1)$, $x_0 = 0$, $x_1 = 0.6$ and $x_2 = 0.9$.
(a) Construct, in two different ways, an interpolation polynomial of degree at most two to approximate f, using the three points (you can use $f(0.6) = 0.47$ and $f(0.9) = 0.6$).

Solution:
Using the Lagrange interpolation formula, we have

$$P(x) = \ln(1)\frac{(x - 0.6)(x - 0.9)}{(0 - 0.6)(0 - 0.9)} + \ln(1.6)\frac{(x - 0)(x - 0.9)}{(0.6 - 0)(0.6 - 0.9)} + \ln(1.9)\frac{(x - 0)(x - 0.6)}{(0.9 - 0)(0.9 - 0.6)},$$

or

$$P(x) = 0 \cdot \frac{(x - 0.6)(x - 0.9)}{(-0.6)(-0.9)} + 0.47\frac{x(x - 0.9)}{(0.6)(-0.3)} + 0.6\frac{x(x - 0.6)}{(0.9)(0.3)}$$

$$P(x) = 0.47\frac{x(x-0.9)}{(0.6)(-0.3)} + 0.6\frac{x(x-0.6)}{(0.9)(0.3)}.$$

Using now Newton's devided differences formula:

$$P(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1),$$

we first compute the coefficients:

$f[x_0] = f(x_0) = 0,$
$f[x_0, x_1] = \frac{f[x_1]-f[x_0]}{x_1-x_0} = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{0.47-0}{0.3} = \frac{0.47}{0.3}.$
$f[x_1, x_2] = \frac{f[x_2]-f[x_1]}{x_2-x_1} = \frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{0.6-0.47}{0.3} = \frac{0.13}{0.3}.$
$f[x_0, x_1, x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0} = \frac{\frac{0.13}{0.3}-\frac{0.47}{0.3}}{0.9} = \frac{-0.34}{0.3\times 0.9}.$
Therefore, $P(x) = 0 + \frac{0.47}{0.3}x + \frac{-0.34}{0.3\times 0.9}x(x-0.6) = \frac{0.47}{0.3}x - \frac{0.34}{0.3\times 0.9}x(x-0.6).$
(the two obtained polynomials should coincide).

(b) Use the Theorem of the course to find an error bound for the approximation.

Solution: We have $n = 2$, thus there is $\xi \in [0, 0.9]$ such that

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)(x - x_2).$$

We have $f'(x) = \frac{1}{x+1}$, $f''(x) = -\frac{1}{(x+1)^2}$, $f'''(x) = (-2)\frac{1}{(x+1)^3}$, with
$|f'''(x)| \leq 2$ for any $x \in [0, 0.9]$.
Thus

$$\begin{aligned}|f(x) - P(x)| &= \frac{|f'''(\xi)|}{6}|(x-0)(x-0.6)(x-0.9)| \\ &\leq \frac{2}{6}0.9 \cdot 0.6 \cdot 0.9 = \frac{1}{3}0.9 \cdot 0.6 \cdot 0.9 = 0.3 \cdot 0.6 \cdot 0.9.\end{aligned}$$

V. Using Taylor's formula, show that

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} - \frac{h}{2}f''(\xi)$$

for some ξ, where $h \neq 0$, $f \in C^2[a, b]$, and $x_0, x_0 + h \in (a, b)$.

Solution: By Taylor's formula with remainder about x_0, we have

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(\xi), \tag{1}$$

for some ξ between x_0 and $x_0 + h$.
Solving for $f'(x_0)$ in (1), we obtain

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} - \frac{h}{2}f''(\xi),$$

thus the desired formula.