Solutions to selected exercises

• Use the Bisection method to find solutions accurate to within $10^{-2}$ for $x^3 - 7x^2 + 14x - 6 = 0$ on $[0,1]$.

Solution: Let $f(x) = x^3 - 7x^2 + 14x - 6 = 0$. Note that $f(0) = -6 < 0$ and $f(1) = 2 > 0$, therefore, based on the Intermediate Value Theorem, since $f$ is continuous, there is $p \in (0,1)$ such that $f(p) = 0$.

Let $a_0 = 0$, $b_0 = 1$, with $f(a_0) < 0$, $f(b_0) > 0$.

Let $p_0 = a_0 + \frac{b_0 - a_0}{2} = 0.5$, and we have $f(p_0) = -0.6250 < 0$ (the same sign as $f(a_0)$), therefore $a_1 = p_0 = 0.5$, $b_1 = b_0 = 1$ and repeat: $p_1 = 0.75$, ...

This yields the following results for $p_n$ and $f(p_n)$:

<table>
<thead>
<tr>
<th>n</th>
<th>p_n</th>
<th>f(p_n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.5</td>
<td>-0.6250000</td>
</tr>
<tr>
<td>1</td>
<td>0.75000000</td>
<td>+0.9843750</td>
</tr>
<tr>
<td>2</td>
<td>0.62500000</td>
<td>+0.2597656</td>
</tr>
<tr>
<td>3</td>
<td>0.56250000</td>
<td>-0.1618652</td>
</tr>
<tr>
<td>4</td>
<td>0.59375000</td>
<td>+0.0540466</td>
</tr>
<tr>
<td>5</td>
<td>0.57812500</td>
<td>-0.0526237</td>
</tr>
<tr>
<td>6</td>
<td>0.58593750</td>
<td>+0.0010313</td>
</tr>
</tbody>
</table>

• Use the theorem from the course to find a bound for the number of iterations needed to achieve an approximation with accuracy $10^{-3}$ to the solution of $x^3 - x - 1 = 0$ lying in the interval $[1,4]$.

Solution: Let’s first verify that $f$ has a zero in the interval $[1,4]$:

$f(1) = -2 < 0$, $f(4) = 64 > 0$, therefore, since $f$ is continuous, by the Intermediate Value Theorem, $f$ has a zero in $[1,4]$.

By the theorem from the course, we impose: $|p_n - p| \leq \frac{b-a}{2^n} = \frac{3}{2^n} \leq 10^{-3}$, then

\[3 \cdot 10^3 \leq 2^n \Rightarrow n \geq \frac{\log_{10}(3 \cdot 10^3)}{\log_{10}(2)} \approx 11.55\]

• Use algebraic manipulation to show that each of the following functions has a fixed point at $p$ precisely when $f(p) = 0$, where $f(x) = x^4 + 2x^2 - x - 3$.

Solution:

(a) $x = g_1(x) \Rightarrow x = (3+x-2x^2)^{1/4} \Rightarrow x^4 = 3+x-2x^2 \Rightarrow x^4+2x^2-x-3 = 0 \Rightarrow f(x) = 0$

(b) $x = g_2(x) \Rightarrow x = \left(\frac{x+3-x^4}{2}\right)^{1/2} \Rightarrow x^2 = \frac{x+3-x^4}{2} \Rightarrow 2x^2 = x + 3 - x^4 \Rightarrow x^4+2x^2-x-3 = 0 \Rightarrow f(x) = 0$
\( (c) \ x = g_3(x) \Rightarrow x = \left( \frac{x^3 + 3}{x^2 + 2} \right)^{1/2} \Rightarrow x^2 = \frac{x^3 + 3}{x^2 + 2} \Rightarrow x^4 + 2x^2 - x - 3 = 0 \Rightarrow f(x) = 0 \quad x^4 + 2x^2 - x - 3 = 0 \Rightarrow f(x) = 0 \)

\( (d) \ x = g_4(x) \Rightarrow x = \frac{3x^4 + 2x^2 + 3}{4x^2 + 2x - 1} \Rightarrow 4x^4 + 4x^2 - x = 3x^4 + 2x^2 + 3 \Rightarrow x^4 + 2x^2 - x - 3 = 0 \Rightarrow f(x) = 0 \)

- \( (a) \) Perform four iterations, if possible, on each of the functions \( g \) defined in Exercise 1. Let \( p_0 = 1 \) and \( p_{n+1} = g(p_n) \) for \( n = 0, 1, 2, 3 \).

\( g_1: \ p_0 = 1, p_1 = 1.1892, p_2 = 1.0801, p_3 = 1.1497, \ f(p_3) = 0.2411 \)
\( g_2: \ p_0 = 1, p_1 = 1.2247, p_2 = 0.9937, p_3 = 1.2286, \ f(p_3) = 1.0688 \)
\( g_3: \ p_0 = 1, p_1 = 1.1547, p_2 = 0.1164, p_3 = 1.1261, \ f(p_3) = 0.0182 \)
\( g_4: \ p_0 = 1, p_1 = 1.1429, p_2 = 0.1245, p_3 = 1.1241, \ f(p_3) = 0.000001 \)

- \( (b) \) Which function do you think gives the best approximation to the solution?

\( g_1 \) gives the best approximation.

- Use the theorem from the course to show that \( g(x) = 2^{-x} \) has a unique fixed point on \([\frac{1}{3}, 1]\). Use a corollary to estimate the number of iterations required to achieve \( 10^{-4} \) accuracy.

**Solution:** We have:
\( g \in C^1 \) on \([\frac{1}{3}, 1]\).
\( g'(x) = -2^{-x} < 0 \Rightarrow g \) is decreasing on \([\frac{1}{3}, 1]\). Then we deduce that if \( \frac{1}{3} \leq x \leq 1 \), then \( 0.7937 = g\left(\frac{1}{3}\right) \geq x \geq g(1) = 0.5 \). Therefore \( g(x) \in \left[\frac{1}{3}, 1\right] \).

We also have that
\[ Max_{x \in [\frac{1}{3}, 1]} |g'(x)| = Max_{x \in [\frac{1}{3}, 1]} 2^{-x} = 2^{-1/3} < 1. \]

Then \( k = 2^{-1/3} = 0.7937 \).

In conclusion, by the theorem from the course, \( g \) has a unique fixed point in \([\frac{1}{3}, 1]\).

Take \( p_0 = 1/3 \) for example. We would like:
\[ |p_n - p| \leq k^n Max\{p_0 - a, b - p_0\} \leq k^n (1 - \frac{1}{3}) \leq 10^{-4}, \]

then \( k^n \leq \frac{10^{-4}}{1 - \frac{1}{3}} \) then \( n \log_{10}(k) \leq \log_{10}\left(\frac{10^{-4}}{1 - \frac{1}{3}}\right) \) or \( n \geq \frac{\log_{10}\left(\frac{10^{-4}}{1 - \frac{1}{3}}\right)}{\log_{10}(k)} \) or \( n \geq 38.10 \)

In practice fewer iterations are needed (this is just an estimate).

(note that \( k = 2^{-1/3} < 1 \) so \( \log_{10}(k) < 0 \)).
• Let \( f(x) = -x^3 - \cos x \) and \( p_0 = -1 \). Use Newton’s method to find \( p_2 \).
Could \( p_0 = 0 \) be used?

\textbf{Solution:} \( f(x) = -x^3 - \cos x \), \( f'(x) = -3x^2 + \sin x \)
Using \( p_0 = -1 \), for \( n \geq 1 \), the Newton’s iteration is: \( p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \).
This gives
\[
\begin{align*}
p_1 &= p_0 - \frac{f(p_0)}{f'(p_0)} \approx -0.8803 \\
p_2 &= p_1 - \frac{f(p_1)}{f'(p_1)} \approx -0.8657 \\
p_0 &= 0 \text{ could not be used because } f'(0) = 0 \text{ (division by 0)}. \\
\end{align*}
\]

• Let \( f(x) = -x^3 - \cos x \). With \( p_0 = -1 \) and \( p_1 = 0 \) find \( p_3 \) by the Secant method.

\textbf{Solution:} The Secant method iteration: \( n \geq 2 \) \( p_{n+1} = p_n - \frac{f(p_n)}{f(p_n)-f(p_{n-1})} \)
\[
\Rightarrow p_{n+1} = p_n - \frac{f(p_n)(p_n-p_{n-1})}{f(p_n)-f(p_{n-1})}.
\]
We have \( f(p_0) = f(-1) = 0.4597 \), \( f(p_1) = f(0) = -1 \), \( p_2 = p_1 - \frac{f(p_1)(p_1-p_0)}{f(p_1)-f(p_0)} \approx -0.6851 \\
p_3 = p_2 - \frac{f(p_2)(p_2-p_1)}{f(p_2)-f(p_1)} \approx -1.2521 \\
\]
Show that the sequence \( p_n = 10^{-2^n} \) converges quadratically to zero.

\textbf{Solution:} We have \( \lim_{n \to \infty} p_n = \lim_{n \to \infty} \frac{1}{10^{2^n}} = 0 \), then, following the definition of quadratic convergent sequence, we compute:
\[
\lim_{n \to \infty} \frac{|p_{n+1} - 0|}{|p_n - 0|^2} = \lim_{n \to \infty} \frac{10^{-2^{n+1}}}{(10^{-2^n})^2} = \lim_{n \to \infty} \frac{10^{-2^{n+1}}}{10^{-2^{2n}}} = 1 < \infty.
\]
Therefore, we have \( \alpha = 2 \) and \( 0 < \lambda = 1 < \infty \) in the definition. This proves that the sequence \( p_n \) converges quadratically to 0.

• Suppose \( p \) is a zero of multiplicity \( m \) of \( f \), where \( f''' \) is continuous on an open interval containing \( p \). Show that the following fixed-point method has \( g'(p) = 0 \):
\[
g(x) = x - m \frac{f(x)}{f'(x)}.
\]

\textbf{Solution:} By the definition of multiplicity of zeros, we have: \( f(x) = (x - p)^m q(x) \), where \( q(p) \neq 0 \). Then
\[
g(x) = x - m \frac{f(x)}{f'(x)} = x - \frac{m(x - p)^m q(x)}{m(x - p)^{m-1} q(x) + (x - p)^m q'(x)} = x - \frac{mq(x) (x - p)}{mq(x) + (x - p) q'(x)}.
\]
Then

\[ g'(x) = 1 - \frac{1}{(mq(x) + (x - p)q'(x))^2} \left[ (mq'(x)(x - p) + mq(x))(mq(x) + (x - p)q'(x)) \\
+ mq(x)(x - p)(mq'(x) + q'(x) + (x - p)q''(x)) \right] \]

\[ \Rightarrow g'(p) = 1 - \frac{(mq(p))^2}{(mq(p))^2} = 0. \]

Remarks on the above problem: note that if \( m = 1 \), then this is Newton’s method. This method can be applied only when we know a-priori the multiplicity \( m \) of the root \( p \). In this case, the method is at least quadratically convergent, because \( g'(p) = 0 \) (see Thm. 2.8).