Solutions to selected exercises

• Use the Bisection method to find solutions accurate to within 10^{-2} for $x^3 - 7x^2 + 14x - 6 = 0$ on [0,1].

Solution: Let $f(x) = x^3 - 7x^2 + 14x - 6 = 0$. Note that $f(0) = -6 < 0$ and $f(1) = 2 > 0$, therefore, based on the Intermediate Value Theorem, since f is continuous, there is $p \in (0, 1)$ such that $f(p) = 0$.

Let $a_0 = 0$, $b_0 = 1$, with $f(a_0) < 0$, $f(b_0) > 0$.

Let $p_0 = a_0 + \frac{b_0 - a_0}{2} = 0.5$, and we have $f(p_0) = -0.6250 < 0$ (the same sign as $f(a_0)$), therefore $a_1 = p_0 = 0.5$, $b_1 = b_0 = 1$ and repeat: $p_1 = 0.75$, ... This yields the following results for p_n and $f(p_n)$:

n	p_n	$f(p_n)$
0	0.5	-0.6250000
1	0.75000000	+0.9843750
2	0.62500000	+0.2597656
3	0.56250000	-0.1618652
4	0.59375000	+0.0540466
5	0.57812500	-0.0526237
6	0.58593750	+0.0010313

• Use the theorem from the course to find a bound for the number of iterations needed to achieve an approximation with accuracy 10^{-3} to the solution of $x^3 - x - 1 = 0$ lying in the interval $[1, 4]$.

Solution: Let's first verify that f has a zero in the interval $[1, 4]$: $f(1) = -2 < 0$, $f(4) = 64 > 0$, therefore, since f is continuous, by the Intermediate Value Theorem, f has a zero in $[1, 4]$.

By the theorem from the course, we impose: $|p_n - p| \leq \frac{b-a}{2^n} = \frac{3}{2^n} \leq 10^{-3}$, then

$$3 \cdot 10^3 \leq 2^n \Rightarrow n \geq \frac{log_{10}(3 \cdot 10^3)}{log_{10}(2)} \approx 11.55$$

• Use algebraic manipulation to show that each of the following functions has a fixed point at p precisely when $f(p) = 0$, where $f(x) = x^4 + 2x^2 - x - 3$.

Solution:

(a) $x = g_1(x) \Rightarrow x = (3+x-2x^2)^{1/4} \Rightarrow x^4 = 3+x-2x^2 \Rightarrow x^4+2x^2-x-3 = 0 \Rightarrow f(x) = 0$

(b) $x = g_2(x) \Rightarrow x = \left(\frac{x+3-x^4}{2}\right)^{1/2} \Rightarrow x^2 = \frac{x+3-x^4}{2} \Rightarrow 2x^2 = x + 3 - x^4 \Rightarrow x^4 + 2x^2 - x - 3 = 0 \Rightarrow f(x) = 0$

(c) $x = g_3(x) \Rightarrow x = \left(\frac{x+3}{x^2+2}\right)^{1/2} \Rightarrow x^2 = \frac{x+3}{x^2+2} \Rightarrow x^4 + 2x^2 = x + 3 \Rightarrow x^4 + 2x^2 - x - 3 = 0 \Rightarrow f(x) = 0$

(d) $x = g_4(x) \Rightarrow x = \frac{3x^4+2x^2+3}{4x^3+4x-1} \Rightarrow 4x^4 + 4x^2 - x = 3x^4 + 2x^2 + 3 \Rightarrow x^4 + 2x^2 - x - 3 = 0 \Rightarrow f(x) = 0$

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(a) Perform four iterations, if possible, on each of the functions g defined in Exercise 1. Let $p_0 = 1$ and $p_{n+1} = g(p_n)$ for $n = 0, 1, 2, 3$.

g_1: $p_0 = 1, p_1 = 1.1892, p_2 = 1.0801, p_3 = 1.1497,\ f(p_3) = 0.2411$

g_2: $p_0 = 1, p_1 = 1.2247, p_2 = 0.9937, p_3 = 1.2286,\ f(p_3) = 1.0688$

g_3: $p_0 = 1, p_1 = 1.1547, p_2 = 0.1164, p_3 = 1.1261,\ f(p_3) = 0.0182$

g_4: $p_0 = 1, p_1 = 1.1429, p_2 = 0.1245, p_3 = 1.1241,\ f(p_3) = 0.000001$

(b) Which function do you think gives the best approximation to the solution ?

g_4 gives the best approximation.

• Use the theorem from the course to show that $g(x) = 2^{-x}$ has a unique fixed point on $[\frac{1}{3}, 1]$. Use a corollary to estimate the number of iterations required to achieve 10^{-4} accuracy.

Solution: We have:

$g \in C^1$ on $[\frac{1}{3}, 1]$.

$g'(x) = -2^{-x} < 0 \Rightarrow g$ is decreasing on $[\frac{1}{3}, 1]$. Then we deduce that if $\frac{1}{3} \leq x \leq 1$, then $0.7937 = g(\frac{1}{3}) \geq x \geq g(1) = 0.5$. Therefore $g(x) \in [\frac{1}{3}, 1]$.

We also have that

$$Max_{x \in [\frac{1}{3},1]} |g'(x)| = Max_{x \in [\frac{1}{3},1]} 2^{-x} = 2^{-1/3} < 1.$$

Then $k = 2^{-1/3} = 0.7937$.

In conclusion, by the theorem from the course, g has a unique fixed point in $[\frac{1}{3}, 1]$

Take $p_0 = 1/3$ for example. We would like:

$$|p_n - p| \leq k^n Max\{p_0 - a, b - p_0\} \leq k^n(1 - \frac{1}{3}) \leq 10^{-4},$$

then $k^n \leq \frac{10^{-4}}{1-\frac{1}{3}}$ then $n \log_{10}(k) \leq \log_{10}\left(\frac{10^{-4}}{1-\frac{1}{3}}\right)$ or $n \geq \frac{\log_{10}\left(\frac{10^{-4}}{1-\frac{1}{3}}\right)}{\log_{10}(k)}$ or $n \geq 38.10$

In practice fewer iterations are needed (this is just an estimate).

(note that $k = 2^{-1/3} < 1$ so $\log_{10}(k) < 0$).

• Let $f(x) = -x^3 - \cos x$ and $p_0 = -1$. Use Newton's method to find p_2. Could $p_0 = 0$ be used ?

Solution: $f(x) = -x^3 - \cos x$, $f'(x) = -3x^2 + \sin x$

Using $p_0 = -1$, for $n \geq 1$, the Newton's iteration is: $p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$. This gives

$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)} \approx -0.8803$

$p_2 = p_1 - \frac{f(p_1)}{f'(p_1)} \approx -0.8657$

$p_0 = 0$ could not be used because $f'(0) = 0$ (division by 0).

• Let $f(x) = -x^3 - \cos x$. With $p_0 = -1$ and $p_1 = 0$ find p_3 by the Secant method.

Solution: The Secant method iteration: $n \geq 2$ $p_{n+1} = p_n - \frac{f(p_n)}{\frac{f(p_n)-f(p_{n-1})}{p_n - p_{n-1}}}$

$\Rightarrow p_{n+1} = p_n - \frac{f(p_n)(p_n - p_{n-1})}{f(p_n)-f(p_{n-1})}$.

We have $f(p_0) = f(-1) = 0.4597$, $f(p_1) = f(0) = -1$, $p_2 = p_1 - \frac{f(p_1)(p_1-p_0)}{f(p_1)-f(p_0)} \approx -0.6851$

$p_3 = p_2 - \frac{f(p_2)(p_2-p_1)}{f(p_2)-f(p_1)} \approx -1.2521$

• Show that the sequence $p_n = 10^{-2^n}$ converges quadratically to zero.

Solution: We have $\lim_{n\to\infty} p_n = \lim_{n\to\infty} \frac{1}{10^{2^n}} = 0$, then, following the definition of quadratic convergent sequence, we compute:

$$\lim_{n\to\infty} \frac{|p_{n+1} - 0|}{|p_n - 0|^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{(10^{-2^n})^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{10^{-2^{n+1}}} = 1 < \infty.$$

Therefore, we have $\alpha = 2$ and $0 < \lambda = 1 < \infty$ in the definition. This proves that the sequence p_n converges quadratically to 0.

• Suppose p is a zero of multiplicity m of f, where f''' is continuous on an open interval containing p. Show that the following fixed-point method has $g'(p) = 0$:

$$g(x) = x - m\frac{f(x)}{f'(x)}.$$

Solution: By the definition of multiplicity of zeros, we have: $f(x) = (x-p)^m q(x)$, where $q(p) \neq 0$. Then

$$g(x) = x - m\frac{f(x)}{f'(x)} = x - \frac{m(x-p)^m q(x)}{m(x-p)^{m-1}q(x) + (x-p)^m q'(x)} = x - \frac{mq(x)(x-p)}{mq(x) + (x-p)q'(x)}.$$

Then

$$g'(x) = 1 - \frac{1}{(mq(x) + (x-p)q'(x))^2}\Big[(mq'(x)(x-p) + mq(x))(mq(x) + (x-p)q'(x))$$
$$+mq(x)(x-p)(mq'(x) + q'(x) + (x-p)q''(x))\Big]$$

$$\Rightarrow\; g'(p) = 1 - \frac{(mq(p))^2}{(mq(p))^2} = 0.$$

Remarks on the above problem: note that if $m = 1$, then this is Newton's method. This method can be applied only when we know a-priori the multiplicity m of the root p. In this case, the method is at least quadratically convergent, because $g'(p) = 0$ (see Thm. 2.8).