Final Exam, Math 151A/2, Winter 2001, UCLA, 03/21/2001, 8am-11am

I.

(a) Let x_0, x_1, ..., x_n be $n+1$ distinct points in $[a, b]$, with $x_0 = a$ and $x_n = b$, and $f \in C^{n+1}[a, b]$. Let $P(x) = P_{0,1,\ldots,n}(x)$ be the Lagrange polynomial interpolating the points x_0, x_1, ..., x_n, such that $P(x_i) = f(x_i)$, for all $i = 0, 1, ..., n$.

Express $f(x)$ in terms of $P(x)$ and a remainder term (the error formula).

(b) Consider the case $n = 2$ and the data

x	$x_0 = 0$	$x_1 = 1$	$x_2 = 2$
$f(x) = \ln(x + 1)$	0	$\ln 2$	$\ln 3$

(i) Find polynomials $L_i(x)$, $i = 0, 1, 2$, such that $L_i(x_i) = 1$ and $L_i(x_j) = 0$ if $i \neq j$.

(ii) Deduce the Lagrange polynomial $P(x) = P_{0,1,2}(x)$ interpolating this data points.

(iii) Write the error formula and find a bound for the error $|f(0.5) - P(0.5)|$.

II.

(a) Let i, j be two distinct integers in $\{0, 1, ..., n\}$. Express $P_{0,1,\ldots,n}(x)$ in terms of $P_{0,1,\ldots,i-1,i+1,\ldots,n}(x)$ and of $P_{0,1,\ldots,j-1,j+1,\ldots,n}(x)$ (Neville's method).

(b) Suppose $x_j = j$ for $j = 0$, 1, 2, 3 and it is known that

$$P_{0,1}(x) = x + 1, \quad P_{1,2}(x) = 3x - 1, \quad \text{and} \quad P_{1,2,3}(1.5) = 4.$$

Find $P_{0,1,2,3}(1.5)$.

III.

For a function f the forward divided-differences are given by

$$\begin{array}{llll} x_0 = 0 & f[x_0] = & & \\ & & f[x_0, x_1] = & \\ x_1 = 1 & f[x_1] = & & f[x_0, x_1, x_2] = 4 \\ & & f[x_1, x_2] = 5 & \\ x_2 = 2 & f[x_2] = 4 & & \end{array}$$

(a) Determine the missing entries in the table.

(b) From (a), express the interpolating Lagrange polynomial $P(x)$ using Newton's forward divided-difference formula for this data points.

IV.

A natural (free) cubic spline S on $[0, 2]$ is defined by

$$S(x) = \begin{cases} S_0(x) = 1 + 2x - x^3, & \text{if } 0 \leq x < 1, \\ S_1(x) = a + b(x-1) + c(x-1)^2 + d(x-1)^3, & \text{if } 1 \leq x \leq 2. \end{cases}$$

Find a, b, c, and d.

V.

Express $f(x_0 + h)$ in terms of $f(x_0)$ and $f'(x_0)$, using the expansion of f in a 1st order Taylor polynomial about x_0 and the remainder term (you should have: $f(x_0 + h) =$ sum of three terms).

Deduce the forward-difference approximation formula for $f'(x_0)$, and express the error term.

VI.

Find the degree of precision of the quadrature formula

$$\int_{-1}^{1} f(x)dx = \frac{1}{3}[f(-1) + 4f(0) + f(1)].$$

VII.

Determine the values of n and h required to approximate

$$\int_0^\pi \sin x dx$$

to within 10^{-5} using the Composite Trapezoidal Rule, and compute the approximation for $n = 4$.

Use the following: $h = (b - a)/n$ and

$$\int_a^b f(x)dx = (quadratureformula) - \frac{b-a}{12}h^2 f''(\mu),$$

from the Composite Trapezoidal Rule, where $\mu \in [a, b]$.

VIII.

Let $A = \begin{bmatrix} 2 & -1 & 3 \\ 2 & 0 & 5 \\ 2 & 1 & 6 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $B = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$.

(a) Use the Gaussian elimination with backward substitution to reduce the linear system $Ax = B$ to an upper triangular system and then solve it completely.

(b) Obtain factorization of the form $A = LU$, where L is lower triangular with 1s on its diagonal, and U is upper triangular. Then solve again the system $Ax = B$ using this factorization.