UCLA MATH 151A, WINTER 2000, FINAL EXAM, MONDAY, MARCH 19

NAME ______________________________ **STUDENT ID #** ____________________

This is a closed-book and closed-note examination. No calculators are allowed. Please show all your work. Partial credit will be given to partial answers. There are 9 problems of total 100 points.

PROBLEM	1	2	3	4	5	6	7	8	9	TOTAL
SCORE										

Problem 1 : Let $[a, b]$ be an interval of $\mathbb{R}$, x_0, x_1, ..., x_n be $n+1$ points in $[a, b]$ and $f \in C^{n+1}(\mathbb{R})$.

1-a Give the form of the Lagrange polynomial of degree at most n such that

$$f(x_k) = P(x_k), \text{ for each } k = 0, 1, ..., n.$$

In the sequel, we will note this polynomial $P_{x_0,x_1,...,x_n}$.

1-b Does another polynomial Q, of degree at most n, exist such that

$$f(x_k) = Q(x_k), \text{ for each } k = 0, 1, ..., n.$$

1-c Deduce from the preceding question that any polynomial of degree at most n is exactly interpolated by a Lagrange polynomial.

1-d State the formula which, for $x \in [a, b]$, expresses $f(x)$ in terms of $P_{x_0,x_1,...,x_n}(x)$ (the Lagrange polynomial defined in question 1-a) and a remainder term.

1-e Deduce from the preceding question that any polynomial of degree at most n is exactly interpolated by a Lagrange polynomial.

1-f Let $(i, j) \in \{0, 1, ..., n\}^2$ be two distinct integers. Taking the notation defined in question 1-a, express $P_{x_0,x_1,...,x_n}$ in terms of $P_{x_0,x_1,...,x_{i-1},x_{i+1},...,x_n}$ and $P_{x_0,x_1,...,x_{j-1},x_{j+1},...,x_n}$.

1-g We can write the Lagrange polynomial $P_{x_0,x_1,...,x_n}$ under the form

$$P_{x_0,x_1,...,x_n}(x) = a_0 + a_1\,(x - x_0) + a_2\,(x - x_0)(x - x_1) + \cdots + a_n\,(x - x_0)(x - x_1)...(x - x_{n-1}).$$

Give the formula which permits to compute the numbers a_i, for $i = 0, 1, ...n$.

1-h Give the three point formula used to approximate $f'(x_0)$ and which involves $f(x_0 - h)$ and $f(x_0 + h)$.

1-i Deduce from the preceding question that $f'(x_0)$ is exactly estimated by

$$\frac{1}{2h}[f(x_0 + h) - f(x_0 - h)]$$

when f is a polynomial of degree 2.

1-j State the Trapezoidal rule for $\int_a^b f(x)dx$.

1-k State the composite Simpson's rule for $\int_a^b f(x)dx$.

Problem 2 : Let $f(x) = \cos(\frac{\pi x}{2})$ and P be the Lagrange polynomial of degree at most 3 which agrees with f at $x_0 = -1$, $x_1 = 0$, $x_2 = 1$, $x_3 = 2$.

2-a Compute $P(0.5)$ using Neville's method.

2-b Express $P(x)$ using Newton's divided difference formula. Use this result to compute $P(0.5)$.

2-c Give a bound for the absolute error $|f(0.5) - P(0.5)|$.

Problem 3 : Let $t > 0$ and $f(x) = e^{tx}$. We call P_n the Lagrange polynomial, of degree at most $n-1$, which agrees with f at the points 1, 2, ...,n.

3-a Check that, for any positive integer n and any $x \in \mathbb{R}$,

$$f^{(n)}(x) = t^n e^{tx} .$$

3-b Give the absolute error $|f(0) - P_n(0)|$ and prove that

$$t^n \leq |f(0) - P_n(0)| \leq t^n e^{tn} .$$

3-c Discuss, in each of the following cases, whether we gain in using large values for n or not when approximating $f(0)$ by $P_n(0)$:

- $t = 2$:
- $t = 0.1$: (Hint : $ln(0.1) \approx -2.3$)

3-d What do you conclude?

Problem 4 : A natural cubic spline S on $[0, 2]$ is defined by

$$S(x) = \begin{cases} S_0(x) = 2 - x - x^2 + 3x^3 & \text{, if } 0 \leq x < 1 \\ S_1(x) = a + b(x-1) + c(x-1)^2 + d(x-1)^3 & \text{, if } 1 \leq x < 2 \end{cases}$$

Find a,b,c,d.

Problem 5 : Let $f(x) = x^3$. You can use the following table to answer the questions

x	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4
$f(x)$	0.216	0.343	0.512	0.729	1	1.331	1.728	2.197	2.744
$\tilde{f}(x)$	0.2	0.3	0.5	0.7	1	1	2	2	3

5-a Use a three points formula, with $h = 0.1$, to compute $f'(1)$.

5-b For this question, you may use the results of the operations listed below.

$$0.512 - 1.728 = -1.216; 1.331 - 0.729 = 0.602; 8 * 0.602 = 4.816.$$

Use a five points formula, with $h = 0.1$, to compute $f'(1)$, and compare your result to the actual value of $f'(1)$. Explain this result.

5-c For this question, you may use the results of the operations listed below.

$$\frac{0.3}{0.2} = 1.5, \frac{1.5}{0.4} = 3.75, \frac{1.7}{0.6} = 2.8333, \frac{2.8}{0.8} = 3.5.$$

Assume that for any $x \in [0,2]$, we only know the approximation $\tilde{f}(x)$ of $f(x)$ (see the bottom line of the table). Use a three points formula, with $h = 0.1$, $h = 0.2$, $h = 0.3$, $h = 0.4$, to compute $f'(1)$.

Which one is the most accurate? Why (answer in one sentence)?

Problem 6 :

6-a State the definition of the degree of precision of a quadrature formula.

6-b Find the constants c_0, c_1, x_1, so that the quadrature formula

$$\int_0^1 f(x)dx = c_0 f(0) + c_1 f(x_1)$$

has the highest possible degree of precision.

Problem 7 : In this exercise, we are going to consider the approximation of

$$\int_0^\pi sin(x)dx$$

by a composite Simpson's rule for n subintervals, with n an even integer.

7-a Find a condition on n, the number of subintervals, such that the composite Simpson's rule approximates $\int_0^\pi sin(x)dx$ with an absolute error less than 0.00001.

7-b Check that $n = 30$ satisfies the condition found in the preceding question. (Hint: $\pi^5 \approx 306$)

7-c Write the algorithm for composite Simpson's rule, in the case of the function $sin(x)$, for a value of $n = 30$.

Problem 8 :

8-a Use the Gaussian elimination with backward substitution to solve the linear system

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & = & 3, \\ 2\,x_1 & - & 3\,x_2 & + & x_3 & = & 0, \\ x_1 & + & 3\,x_2 & - & 3\,x_3 & = & 1. \end{array}$$

8-b Use the Gaussian elimination with backward substitution to solve the following linear system for a parameter $e \in \mathbb{R}$.

$$\begin{array}{ccccc} x_1 & - & 4\,x_2 & = & -3, \\ (1+e)\,x_1 & - & 4.1\,x_2 & = & -3.1. \end{array}$$

8-c Interpreting e as a roudoff error, what do you think about the stability of the result (x_1, x_2) with regard to variations of e.

Problem 9 :

9-a Use Newton-Raphson method to find a function g such that

$$g(\sqrt{3}) = \sqrt{3}\,.$$

9-b Prove that, in general, given $[a, b]$ an interval and $g \in C^1[a, b]$ such that g has a unique fixed point p in $[a, b]$ and such that, there exists $0 < k < 1$, with

$$|g'(x)| \leq k, \text{ for all } x \in (a, b),$$

we have

$$|p_n - p| \leq k^n \, max\{|p_0 - a|, |p_0 - b|\} .$$

(Hint: Prove first that $|p_n - p| \leq k|p_{n-1} - p|$.)

9-c Find an interval (a, b), an initial point $p_0 \in (a, b)$ and a condition on the number of iteration $n \in \mathbb{N}$ such that the fixed point algorithm applied to the function of question 9-a approximates $\sqrt{3}$ with the accuracy 10^{-3}.

Hint :$\sqrt{3} \in [\frac{3}{2}, 2]$.

9-d Write Newton's algorithm for the function and the initial point you determined previously, for n iterations.