This is a closed-book and closed-note examination. No calculators are allowed. Please show all your work. Partial credit will be given to partial answers. There are 9 problems of total 100 points.

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**Problem 1**: Let \([a, b]\) be an interval of \(\mathbb{R}\), \(x_0, x_1, ..., x_n\) be \(n+1\) points in \([a, b]\) and \(f \in C^{n+1}(\mathbb{R})\).

1-a Give the form of the Lagrange polynomial of degree at most \(n\) such that

\[ f(x_k) = P(x_k), \text{ for each } k = 0, 1, ..., n. \]

In the sequel, we will note this polynomial \(P_{x_0, x_1, ..., x_n}\).

1-b Does another polynomial \(Q\), of degree at most \(n\), exist such that

\[ f(x_k) = Q(x_k), \text{ for each } k = 0, 1, ..., n. \]

1-c Deduce from the preceding question that any polynomial of degree at most \(n\) is exactly interpolated by a Lagrange polynomial.

1-d State the formula which, for \(x \in [a, b]\), expresses \(f(x)\) in terms of \(P_{x_0, x_1, ..., x_n}(x)\) (the Lagrange polynomial defined in question 1-a) and a remainder term.

1-e Deduce from the preceding question that any polynomial of degree at most \(n\) is exactly interpolated by a Lagrange polynomial.

1-f Let \((i, j) \in \{0, 1, ..., n\}^2\) be two distinct integers. Taking the notation defined in question 1-a, express \(P_{x_0, x_1, ..., x_n}\) in terms of \(P_{x_0, x_1, ..., x_{i-1}, x_{i+1}, ..., x_n}\) and \(P_{x_0, x_1, ..., x_{j-1}, x_{j+1}, ..., x_n}\).

1-g We can write the Lagrange polynomial \(P_{x_0, x_1, ..., x_n}\) under the form

\[ P_{x_0, x_1, ..., x_n}(x) = a_0 + a_1 (x - x_0) + a_2 (x - x_0)(x - x_1) + \cdots + a_n (x - x_0)(x - x_1)...(x - x_{n-1}). \]

Give the formula which permits to compute the numbers \(a_i\), for \(i = 0, 1, ..., n\).

1-h Give the three point formula used to approximate \(f'(x_0)\) and which involves \(f(x_0 - h)\) and \(f(x_0 + h)\).

1-i Deduce from the preceding question that \(f'(x_0)\) is exactly estimated by

\[ \frac{1}{2h}[f(x_0 + h) - f(x_0 - h)] \]

when \(f\) is a polynomial of degree 2.

1-j State the Trapezoidal rule for \(\int_a^b f(x)dx\).

1-k State the composite Simpson’s rule for \(\int_a^b f(x)dx\).
Problem 2: Let \( f(x) = \cos \left( \frac{\pi x}{2} \right) \) and \( P \) be the Lagrange polynomial of degree at most 3 which agrees with \( f \) at \( x_0 = -1, x_1 = 0, x_2 = 1, x_3 = 2.\)

2-a Compute \( P(0.5) \) using Neville’s method.

2-b Express \( P(x) \) using Newton’s divided difference formula. Use this result to compute \( P(0.5). \)

2-c Give a bound for the absolute error \(|f(0.5) - P(0.5)|.\)

Problem 3: Let \( t > 0 \) and \( f(x) = e^{tx}. \) We call \( P_n \) the Lagrange polynomial, of degree at most \( n-1, \) which agrees with \( f \) at the points 1, 2, ..., \( n.\)

3-a Check that, for any positive integer \( n \) and any \( x \in \mathbb{R}, \)
\[
f^{(n)}(x) = t^n e^{tx}.
\]

3-b Give the absolute error \(|f(0) - P_n(0)|\) and prove that
\[
t^n \leq |f(0) - P_n(0)| \leq t^n e^{tn}.
\]

3-c Discuss, in each of the following cases, whether we gain in using large values for \( n \) or not when approximating \( f(0) \) by \( P_n(0):\)

- \( t = 2: \)
- \( t = 0.1: \) (Hint: \( \ln(0.1) \approx -2.3)\)

3-d What do you conclude?

Problem 4: A natural cubic spline \( S \) on [0, 2] is defined by
\[
S(x) = \begin{cases} 
S_0(x) = 2 - x - x^2 + 3x^3, & \text{if } 0 \leq x < 1 \\
S_1(x) = a + b(x - 1) + c(x - 1)^2 + d(x - 1)^3, & \text{if } 1 \leq x < 2
\end{cases}
\]

Find \( a, b, c, d. \)

Problem 5: Let \( f(x) = x^3. \) You can use the following table to answer the questions

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1</th>
<th>1.1</th>
<th>1.2</th>
<th>1.3</th>
<th>1.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>0.216</td>
<td>0.343</td>
<td>0.512</td>
<td>0.729</td>
<td>1</td>
<td>1.331</td>
<td>1.728</td>
<td>2.197</td>
<td>2.744</td>
</tr>
<tr>
<td>( f(x) )</td>
<td>0.2</td>
<td>0.3</td>
<td>0.5</td>
<td>0.7</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
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5-a Use a three points formula, with \( h = 0.1, \) to compute \( f'(1). \)

5-b For this question, you may use the results of the operations listed below.
\[
0.512 - 1.728 = -1.216; 1.331 - 0.729 = 0.602; 8 \times 0.602 = 4.816.
\]

Use a five points formula, with \( h = 0.1, \) to compute \( f'(1), \) and compare your result to the actual value of \( f'(1). \) Explain this result.
5-c For this question, you may use the results of the operations listed below.

\[
\begin{array}{cccc}
0.3 & 1.5 & 3.75 & 2.8333 \\
0.2 & 0.4 & 0.6 & 0.8 \\
\end{array}
\]

Assume that for any \(x \in [0, 2]\), we only know the approximation \(\tilde{f}(x)\) of \(f(x)\) (see the bottom line of the table). Use a three points formula, with \(h = 0.1, h = 0.2, h = 0.3, h = 0.4\), to compute \(f'(1)\). Which one is the most accurate? Why (answer in one sentence)?

**Problem 6 :**

6-a State the definition of the degree of precision of a quadrature formula.

6-b Find the constants \(c_0, c_1, x_1\), so that the quadrature formula

\[
\int_0^1 f(x)dx = c_0 f(0) + c_1 f(x_1)
\]

has the highest possible degree of precision.

**Problem 7 :** In this exercise, we are going to consider the approximation of

\[
\int_0^\pi \sin(x)dx
\]

by a composite Simpson’s rule for \(n\) subintervals, with \(n\) an even integer.

7-a Find a condition on \(n\), the number of subintervals, such that the composite Simpson’s rule approximates \(\int_0^\pi \sin(x)dx\) with an absolute error less than 0.00001.

7-b Check that \(n = 30\) satisfies the condition found in the preceding question. (Hint: \(\pi^5 \approx 306\))

7-c Write the algorithm for composite Simpson’s rule, in the case of the function \(\sin(x)\), for a value of \(n = 30\).

**Problem 8 :**

8-a Use the Gaussian elimination with backward substitution to solve the linear system

\[
\begin{align*}
2x_1 - 3x_2 + x_3 &= 0, \\
x_1 + 3x_2 - 3x_3 &= 1.
\end{align*}
\]

8-b Use the Gaussian elimination with backward substitution to solve the following linear system for a parameter \(e \in \mathbb{R}\).

\[
\begin{align*}
(x + e)x_1 - 4x_2 &= -3, \\
(1 + e)x_1 - 4.1x_2 &= -3.1.
\end{align*}
\]

8-c Interpreting \(e\) as a roundoff error, what do you think about the stability of the result \((x_1, x_2)\) with regard to variations of \(e\).

**Problem 9 :**

9-a Use Newton-Raphson method to find a function \(g\) such that

\[g(\sqrt{3}) = \sqrt{3} \]
9-b Prove that, in general, given \([a, b]\) an interval and \(g \in C^1[a, b]\) such that \(g\) has a unique fixed point \(p\) in \([a, b]\) and such that, there exists \(0 < k < 1\), with
\[
|g'(x)| \leq k, \text{ for all } x \in (a, b),
\]
we have
\[
|p_n - p| \leq k^n \max\{|p_0 - a|, |p_0 - b|\}.
\]
(Hint: Prove first that \(|p_n - p| \leq k|p_{n-1} - p|\).)

9-c Find an interval \((a, b)\), an initial point \(p_0 \in (a, b)\) and a condition on the number of iteration \(n \in \mathbb{N}\) such that the fixed point algorithm applied to the function of question 9-a approximates \(\sqrt{3}\) with the accuracy \(10^{-3}\).
Hint : \(\sqrt{3} \in [\frac{3}{2}, 2]\).

9-d Write Newton’s algorithm for the function and the initial point you determined previously, for \(n\) iterations.