MATH 115A/3, Spring 2005, Midterm #1
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Solutions

QUESTION	SCORE
[1]	18
[2]	15
[3]	15
[4]	19
[5]	15
[6]	18
TOTAL	100

Class average: 51.5

[1] Consider the vector space $V = \mathcal{M}_{3\times 3}(F)$ of all 3×3 matrices with entries from a field F. Let W be the subset of V of diagonal matrices:

$$W = \Big\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad a, b, c \in F \Big\}.$$

Show that W is a subspace of V and give an example of a basis of W.

Solution: To show that W is a subspace of V, we need to show that $0_V \in W$, and that W is close over the vector addition and scalar multiplication.

For $a = b = c = 0_F$, we have $0_V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in W$.

For $x, y \in W$, with $x = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$, $y = \begin{pmatrix} d & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & f \end{pmatrix}$, where $a, b, c, d, e, f \in F$, we have $x + y = \begin{pmatrix} a+d & 0 & 0 \\ 0 & b+e & 0 \\ 0 & 0 & c+f \end{pmatrix} \in W$. Similarly, if $d \in F$, then $dx = d\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = \begin{pmatrix} da & 0 & 0 \\ 0 & db & 0 \\ 0 & 0 & dc \end{pmatrix} \in W$, because $d0_F = 0_F$ for any $d \in F$. Therefore, W is a subspace of V.

A basis of W is for instance

$$\beta = \{E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\},$$

and any $x \in W$ as above satisfies $x = aE_1 + bE_2 + cE_3$, therefore β is a generator of W.

To show that $\beta = l.i.$, assume that $\alpha E_1 + \beta E_2 + \gamma E_3 = 0_V$, for some scalars $\alpha, \beta, \gamma \in F$. This implies that $\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, therefore $\alpha = \beta = \gamma = 0_F$.

We conclude that β is a linearly independent set. Since it is also a generating set for W, then β is a basis of W.

[2] Let V be a vector space of dimension 3 over a field F, and let $\beta = \{v_1,\ v_2,\ v_3\}$ be a basis of V. Prove that every vector $v \in V$ can be expressed uniquely as a linear combination of v_1, v_2, and v_3.

Solution: By the definition of the basis, we know that β is a generator of V, therefore any $v \in V$ can be expressed as a linear combination

$$v = c_1 v_1 + c_2 v_2 + c_3 v_3,$$

with some scalars $c_1, c_2, c_3 \in F$. Assume that v is also given by another linear combination of β:

$$v = d_1 v_1 + d_2 v_2 + d_3 v_3,$$

for scalars $d_1, d_2, d_3 \in F$.

Then

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = d_1 v_1 + d_2 v_2 + d_3 v_3,$$

or

$$(c_1 - d_1)v_1 + (c_2 - d_2)v_2 + (c_3 - d_3)v_3 = 0_V.$$

Since β is a basis, then by definition $\beta = \{v_1, v_2, v_3\}$ is linearly independent. We then deduce that $c_1 - d_1 = 0_F$, $c_2 - d_2 = 0_F$, $c_3 - d_3 = 0_F$, or that $c_1 = d_1$, $c_2 = d_2$, $c_3 = d_3$. Therefore, any $v \in V$ can be expressed uniquely as a linear combination of vectors v_1, v_2, v_3.

[3] Let V and W be vector spaces over the same field F, and let $T : V \to W$ be a linear transformation. Let $v_1, ..., v_k \in V$, and suppose that $\{T(v_1), T(v_2), ..., T(v_k)\}$ are linearly independent. Prove that $v_1, v_2, ..., v_k$ are linearly independent.

Solution: Assume that there are scalars $\alpha_1, ..., \alpha_k \in F$ such that

$$\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_k v_k = 0_V. \tag{1}$$

Since T is linear, we have $T(0_V) = 0_W$, therefore

$$T(\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_k v_k) = T(0_V) = 0_W.$$

Again by the linearity of T we deduce

$$\alpha_1 T(v_1) + \alpha_2 T(v_2) + ... + \alpha_k T(v_k) = 0_W.$$

Now, since $\{T(v_1), T(v_2), ..., T(v_k)\}$ are linearly independent, we must have

$$\alpha_1 = \alpha_2 = ... = \alpha_k = 0_F. \tag{2}$$

From (1) and (2), we deduce that $\{v_1, v_2, ..., v_k\}$ must be linearly independent.

[4] Construct a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$, such that

$$T(1,0) = (1, -1, 2) \text{ and } T(0,1) = (2, 1, -2).$$

Find $T(3,5)$ and the null space $N(T)$.

What is the dimension of the range space $R(T)$?

Solution: Note that $\{e_1 = (1,0),\ e_2 = (0,1)\}$ forms a basis for $\mathbb{R}^2$ (the standard canonical basis), therefore any $(x,y) \in \mathbb{R}^2$ can be expressed as

$$(x,y) = x(1,0) + y(0,1) = xe_1 + ye_2.$$

Define $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(x,y) = xT(1,0) + yT(0,1) = x(1,-1,2) + y(2,1,-2) = (x + 2y, -x + y, 2x - 2y)$.

T is linear by construction: indeed, if $u = (x_1, y_1) \in \mathbb{R}^2$, $v = (x_2, y_2) \in \mathbb{R}^2$, and $c \in \mathbb{R}$, then $T(cu + v) = T\big((cx_1 + x_2)(1,0) + (cy_1 + y_2)(0,1)\big) = (cx_1 + x_2)T(1,0) + (cy_1 + y_2)T(0,1) = c(x_1 T(1,0) + y_1 T(0,1)) + x_2 T(1,0) + y_2 T(0,1) = cT(u) + T(v)$.

We have $T(3,5) = 3T(1,0) + 5T(0,1) = 3(1,-1,2) + 5(2,1,-2) = (13, 2, -4) \in \mathbb{R}^3$ by the definition of T or the linearity.

Recall $N(T) = \{(x,y) \in I\!R^2 : T(x,y) = (0,0,0)\}$. Assume $(x,y) \in N(T)$. Then $T(x,y) = (x+2y, -x+y, 2x-2y) = (0,0,0)$, therefore

$$x + 2y = 0, \ -x + y = 0, \ 2x - 2y = 0.$$

This implies that $x = y = 0$ only, and $N(T) = \{(0,0) = 0_{I\!R^2}\}$. By the dimension theorem, we have $dim(N(T)) + dim(R(T)) = dim(I\!R^2) = 2$, and $dim(N(T)) = 0$, therefore $dim(R(T)) = 2$.

[5] Consider $V = P_5(F)$ the vector space of polynomials of degree at most 5 on a field F, with coefficients from F.

(a) Is $S = \{p \in P_5(F), p(x) = 2 + ax + bx^3\}$ a subspace of V ?

(b) Give an example of a subspace W of V such that dim(W)=3. Justify your answers.

Solution:

(a) Recall that 0_V is the polynomial identically equal with zero (with all zero coefficients). Clearly the polynomial 0_V does not belong to S (because all polynomials in S have the coefficient 2 of x^0), therefore S cannot be a subspace.

(b) Take $W = \{p \in P_5(F) : \ p(x) = a + bx + cx^3, \ a, b, c \in F\}$. Then W is a subspace of V and $dimW = 3$. Indeed:

For $a = b = c = 0_F$, $p(x) = a + bx + cx^3 = 0_V \in W$, therefore W contains the zero element of W.

For $p_1, p_2 \in W$ with $p_1(x) = a_1 + b_1x + c_1x^3$, $p_2(x) = a_2 + b_2x + c_2x^3$, we have $(p_1+p_2)(x) = (a_1+a_2)+(b_1+b_2)x+(c_1+c_2)x^3 \in W$, and $dp_1(x) = (da_1)+(db_1)x+(dc_1)x^3 \in W$, for any scalar $d \in W$. Therefore, W is a subspace of V.

Note that $\beta = \{e_1(x) = 1_F, e_2(x) = x, e_3(x) = x^3\}$ is a basis of W. Indeed, β obviously generates W: any $p \in W$ with $p(x) = a+bx+cx^3 = ae_1(x)+be_2(x)+ce_3(x)$. Also, β is l.i., because if $ae_1(x) + be_2(x) + ce_3(x) = 0$ for any $x \in F$, then $a = b = c = 0_F$.

In conclusion, β is a basis of W and this gives that $dimW = 3$, given that β has 3 elements.

[6] Find a basis for the following subspace of $I\!R^5$:

$$W = \{(a_1, a_2, a_3, a_4, a_5) \in I\!R^5 : \ a_1 - a_3 - a_4 = 0\}.$$

What is the dimension of W ?

Solution: Let $a = (a_1, a_2, a_3, a_4, a_5) \in W$. Then $a_1 = a_3 + a_4$, or

$$a = (a_3 + a_4, a_2, a_3, a_4, a_5) \in W.$$

Let $\{e_1, e_2, e_3, e_4, e_5\}$ be the standard canonical basis of $I\!R^5$.

Then $a = (a_3 + a_4, a_2, a_3, a_4, a_5) = (a_3 + a_4)e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 = a_2e_2 + a_3(e_1+e_3) + a_4(e_1+e_4) + a_5e_5$. This shows that $\beta = \{e_2, \ e_1+e_3, \ e_1+e_4, \ e_5\}$ is a generator for W.

Clearly β is l.i. Indeed, if $c_1e_2 + c_2(e_1+e_3) + c_3(e_1+e_4) + c_4e_5 = 0_V$ for some scalars c_i, then $(c_2+c_3)e_1 + c_1e_2 + c_2e_3 + c_3e_4 + c_4e_5 = 0_V$, but $\{e_1, e_2, e_3, e_4, e_5\}$ are l.i., then $c_2 + c_3 = 0$, $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$, i.e. $c_i = 0$, $i = 1, ..., 4$.

In conclusion $\beta = \{(0,1,0,0,0), (1,0,1,0,0), (1,0,0,1,0), (0,0,0,0,1)\}$ is a basis of W and $dimW = 4$.