• Justify your answers.
• Calculators are not allowed.
• This is a closed-book and closed-note exam.

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[1] Consider the vector space $V = F^5$, over a field $F$. Show that  

\[ W = \{ (a_1, a_2, a_3, a_4, a_5) \in F^5 : \ a_2 = a_3 = a_4 \text{ and } a_1 + a_5 = 0 \} \]

is a subspace of $V$. Find its dimension and give an example of a basis.

Justify your answers.

**Solution:** To show that $W$ is a subspace:

- $0_V = (0, 0, 0, 0, 0) \in W$ since obviously it satisfies $a_2 = a_3 = a_4 (= 0)$ and $a_1 + a_5 = 0 + 0 = 0$.

- Let arbitrary $u = (a_1, a_2, a_3, a_4, a_5) = (a_1, a_2, a_2, a_2, -a_1) \in W$ and $v = (b_1, b_2, b_3, b_4, b_5) = (b_1, b_2, b_2, -b_1) \in W$, since $a_2 = a_3 = a_4$, $a_1 + a_5 = 0$, and $b_2 = b_3 = b_4, b_1 = b_5$.

Then $u + v = (a_1 + b_1, a_2 + b_2, a_2 + b_2, a_2 + b_2, -(a_1 + b_2)$ and clearly $u + v \in W$.

- Let $c \in F$ arbitrary and $u \in W$ as above. Then $cu = (ca_1, ca_2, ca_3, ca_4, ca_5) = (ca_1, ca_2, ca_2, ca_2, c(-a_1)) = (ca_1, ca_2, ca_2, ca_2, -ca_1) \in W$ obviously.

Thus $W$ contains the zero element from $V$, and it is closed over the vector addition and scalar multiplication. Therefore, by a thm. from the course, $W$ is a subspace of $V$.

Any vector $u \in W$ can be expressed as $u = (a_1, a_2, a_3, a_4, a_5) = (a_1, a_2, a_2, a_2, -a_1) = a_1(1, 0, 0, 0, -1) + a_2(0, 1, 1, 1, 0)$, with arbitrary $a_1 \in F, a_2 \in F$. Therefore the subset of $W$, $\beta = \{ u_1 = (1, 0, 0, 0, -1), u_2 = (0, 1, 1, 1, 0) \}$ is a generator for $W$, or $\text{Span}(\beta) = W$.

In addition, the vectors $u_1$ and $u_2$ are linearly independent (can be easily verified), thus by the definition of a basis, $\beta$ must be a basis for $W$ and we deduce that $\dim(W)=2$.

[2] Let $T : \mathcal{M}^{2\times3}(F) \mapsto \mathcal{M}^{2\times2}(F)$ be defined by

\[ T \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix} \]

Show that $T$ is a linear transformation, and find bases for both $N(T)$ and $R(T)$. Then compute the nullity and rank of $T$, and verify the dimension theorem. Determine whether $T$ is one-to-one or onto, using appropriate theorems.

**Solution:**

To show that $T$ is linear: let arbitrary $c \in F$ and $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$,

\[ B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}, \quad A, B \in \mathcal{M}^{2\times3}(F) \]

\[ T(cA + B) = T \begin{pmatrix} ca_{11} + b_{11} & ca_{12} + b_{12} & ca_{13} + b_{13} \\ ca_{21} + b_{21} & ca_{22} + b_{22} & ca_{23} + b_{23} \end{pmatrix} \]

\[ = \begin{pmatrix} 2(ca_{11} + b_{11}) - (ca_{12} + b_{12}) & (ca_{13} + b_{13}) + 2(ca_{12} + b_{12}) \\ 0 & 0 \end{pmatrix} = cT(A) + T(B), \]

thus $T$ is linear (no need to verify that $T(O) = O$).

Let $A \in N(T)$ as above, this means $T(A) = O$ or $2a_{11} - a_{12} = 0$, 

\[ 2a_{11} - a_{12} = 0 \]

\[ \implies \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \]

\[ \implies A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \]

Therefore, $N(T) = \{ A \in \mathcal{M}^{2\times3}(F) : 2a_{11} - a_{12} = 0 \}$.
\[ a_{13} + 2a_{12} = 0. \]

Therefore \( a_{12} = 2a_{11}, a_{13} = -4a_{11}. \) Thus \( A \in N(T) \) iff \( A = \begin{pmatrix} a_{11} & 2a_{11} & -4a_{11} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 1 & 2 & -4 \\ 0 & 0 & 0 \end{pmatrix} + a_{21} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + a_{22} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + a_{23} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \) Therefore the set \( \{ \begin{pmatrix} 1 & 2 & -4 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \} \) is a generator of \( N(T). \) Moreover, the four matrices of this set are linearly independent (easy to see), thus this set forms a basis for \( N(T) \) and \( \dim N(T) = 4. \)

A basis for \( R(T) \) clearly is \( \{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \} \) (lin. indep. and generator of \( R(T) \)). Thus \( \dim R(T) = 2. \)

We can verify Dim. Thm. which states \( \dim (V) = \dim (N(T)) + \dim (R(T)), \) or here \( 6 = 4 + 2, \) thus satisfied.

Finally, \( T \) is not 1-to-1 because \( \dim N(T) = 4 \neq 0. \) Also, \( T \) is not onto because \( \dim R(T) = 2 
eq 4 = \dim M^{2 \times 2}(F), \) thus \( R(T) \subset M^{2 \times 2}(F) \) with strict inclusion.

[3] Let \( u, v \) and \( w \) be distinct vectors of a vector space \( V. \) Show that if \( \{u, v, w\} \) is a basis for \( V, \) then \( \{u + v + w, v + w, w\} \) is also a basis for \( V. \)

**Solution:** Clearly \( \dim V = 3. \) By a theorem from the course, it is sufficient to show that the set \( \{u + v + w, v + w, w\} \) is linearly independent, since it contains exactly three elements and the dim. of \( V = 3. \)

Assume that
\[ \alpha_1(u + v + w) + \alpha_2(v + w) + \alpha_3w = 0_V \tag{1} \]
for some scalars \( \alpha_1, \alpha_2, \alpha_3 \in F. \)

Rearranging the terms, we deduce \( \alpha_1u + (\alpha_1 + \alpha_2)v + (\alpha_1 + \alpha_2 + \alpha_3)w = 0_V. \)
Since \( \{u, v, w\} \) is linearly independent (being a basis of \( V), \) we deduce that
\[ \alpha_1 = 0_F, \]
\[ \alpha_1 + \alpha_2 = 0_F, \]
\[ \alpha_1 + \alpha_2 + \alpha_3 = 0_F. \]

Therefore, from the first relation we get \( \alpha_1 = 0, \) using this in the second we get \( 0 + \alpha_2 = 0 \) thus \( \alpha_2 = 0. \) With \( \alpha_1 = \alpha_2 = 0 \) in the third relation, we get \( 0 + 0 + \alpha_3 = 0, \) therefore also \( \alpha_3 = 0. \)

In conclusion, since (1) implies \( \alpha_1 = \alpha_2 = \alpha_3 = 0, \) we deduce that \( \{u + v + w, v + w, w\} \) is linearly independent, and this concludes the proof.

[4] Let \( V \) and \( W \) be vector spaces and \( T : V \rightarrow W \) be linear.
(a) Suppose that \( T \) is one-to-one and that \( S = \{v_1, ..., v_k\} \) is a subset of \( V. \)
Prove that \( S \) is linearly independent if and only if \( T(S) = \{T(v_1), ..., T(v_k)\} \) is linearly independent.
(b) Suppose \( \beta = \{v_1, ..., v_n\} \) is a basis for \( V \) and \( T \) is one-to-one and onto. Prove that \( T(\beta) = \{T(v_1), ..., T(v_n)\} \) is a basis for \( W. \)

**Solution:**
(a) \( (\Rightarrow) \) Assume \( S \) linearly independent.
Let \( a_1, ..., a_k \in F \) such that
\[ a_1T(v_1) + a_2T(v_2) + ... + a_kT(v_k) = 0_W. \tag{2} \]
Since $T$ is linear, this is equivalent with

$$T(a_1v_1 + a_2v_2 + \ldots + a_kv_k) = 0_V.$$  

Moreover, since $T$ is 1-to-1, we know that $N(T) = \{0\}$ (only $0_V$ goes to $0_W$ through $T$), thus we obtain $a_1v_1 + a_2v_2 + \ldots + a_kv_k = 0_V$. Now, since $S$ is linearly independent, we must have $a_1 = a_2 = \ldots = a_k = 0$.

In conclusion, since (2) implies $a_1 = a_2 = \ldots = a_k = 0_F$, the set $T(S)$ must also be linearly independent.

$(\Leftarrow)$ Assume that $T(S)$ is lin. ind.

Let $a_1, \ldots, a_k \in F$ such that

$$a_1v_1 + a_2v_2 + \ldots + a_kv_k = 0_V. \quad (3)$$

Applying $T$ to both sides, and using $T(0_V) = 0_W$, we get

$$T(a_1v_1 + a_2v_2 + \ldots + a_kv_k) = T(0_V) = 0_W,$$

or, by linearity

$$a_1T(v_1) + a_2T(v_2) + \ldots + a_kT(v_k) = 0_W.$$  

Now, since $T(S)$ is linearly independent, we must have $a_1 = a_2 = \ldots = a_k = 0_F$.

In conclusion, since (3) implies $a_1 = a_2 = \ldots = a_k = 0_F$, the set $S$ must also be linearly independent.

[5] (a) Construct a linear transformation $T : R^2 \mapsto R^3$ such that $T(1, 1) = (1, 0, 2)$ and $T(2, 3) = (1, -1, 4)$. What is $T(8, 11)$? Is $T$ onto? Explain.

(b) Is there a linear transformation $T : R^3 \mapsto R^2$ such that $T(1, 0, 3) = (1, 1)$ and $T(-2, 0, -6) = (2, 1)$? Explain.

**Solution:**

(a) Note that $\{u_1 = (1, 1), u_2 = (2, 3)\}$ is linearly independent, therefore a basis of $R^2$ (easy to verify), thus such transformation exist and is unique.

Any vector $x = (x_1, x_2) = (3x_1 - 2x_2)(1, 1) + (x_2 - x_1)(2, 3)$.

Therefore we define $T$ by $T(x) = (3x_1 - 2x_2)T(1, 1) + (x_2 - x_1)T(2, 3) = (3x_1 - 2x_2)(1, 0, 2) + (x_2 - x_1)(1, -1, 4)$. We obtain $T(x_1, x_2) = (2x_1 - x_2, x_1 - x_2, 2x_1)$. By a theorem from the course, $T$ is linear (or this can be verified directly).

Thus $T(8, 11) = (5, -3, 16)$.

$T$ cannot be onto: since by dim. thm. $\text{dim}(N(T)) + \text{dim}(R(T)) = 2$, therefore $\text{dim}(R(T)) \leq 2 \neq 3$, thus $R(T) \neq R^3$ and $T$ is not onto.

(b) Such $T$ does not exist, because it would not satisfy the definition of a linear transformation: indeed,

$$T(-2, 0, -6) = T(-2(1, 0, 3)) = (2, 1) \neq -2T(1, 0, 3) = (-2, -2)$$

thus does not satisfy $T(cu) = cT(u)$ for scalar $c = -2$ and vector $(1, 0, 3)$.

Many students gave the incorrect answer: “$T$ is not linear because $(1, 0, 3)$ and $(-2, 0, -6)$ are not lin. indep.” (why incorrect? the converse of Thm. 2.6 does not hold, or by other reasons: linear transformations are defined for any vectors of $V$, linear dep. or lin. indep.)