Solutions to selected exercises from the textbook

Exercise 2/2.1: $T : R^3 \to R^2$ defined by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$. To find $N(T)$ and a basis, let $a_1, a_2, a_3$ be s.t. $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3) = (0, 0)$. Then
\[
\begin{align*}
\{ & a_1 - a_2 = 0 \\
2a_3 &= 0
\end{align*}
\]
or $a_1 = a_2$, $a_3 = 0$. Therefore $N(T) = \{(a, a, 0), a \in R\}$ and $\text{dim}(N(T)) = 1$, a basis of $N(T)$ is given by $\{(1, 1, 0)\}$, therefore $\text{nullity}(T) = 1$. We deduce (by Thm. 2.4) that $T$ is not one-to-one, because $N(T) \neq \{0\}$.

We could have obtained this statement directly by the definition: note that there are distinct vectors $\vec{a}, \vec{b}$ in $R^3$ such that $T(\vec{a}) = T(\vec{b})$ with $\vec{a} \neq \vec{b}$. Indeed $T(2, 2, 1) = T(1, 1, 1) = (0, 2)$, therefore $T$ is not one-to-one.

By the Dimension Theorem, we obtained that $\text{dim}(R(T)) = 3 - \text{dim}(N(T)) = 2$. This implies that dim. of range of $T$ coincides with dim. of $R^2$, i.e. $R(T) = R^2$, i.e. $T$ is onto.

This could have been done in the following way: let $(x, y) \in R^2$ be arbitrary, and find $(a_1, a_2, a_3)$, if any, such that $T(a_1, a_2, a_3) = (x, y)$. This would imply $a_1 - a_2 = x$, and $2a_3 = y$. We see that $(x, y)$ is always in the image of $T$, by $T(a, a - x, y/2) = (x, y)$. Again, we conclude that $T$ is onto and a basis of $R(T)$ is any basis of $R^2$, for instance of standard basis $\{(1, 0), (0, 1)\}$.

Exercise 5/2.1: $T : P_2(R) \to P_2(R)$, $T(f(x)) = xf(x) + f'(x)$.

Note that $T\left(a_0 + a_1x + a_2x^2\right) = a_0x + a_1x^2 + a_2x^3 + (a_1 + 2a_2)x = a_1 + (a_0 + 2a_2)x + a_1x^2 + a_2x^3$, with $f(x) = a_0 + a_1x + a_2x^2$.

$f(x) \in N(T)$ if $T(f(x)) = 0$ for any $x$, therefore if $a_1 = 0$, $a_0 + 2a_2 = 0$, $a_1 = 0$, $a_2 = 0$, i.e. $a_0 = a_1 = a_2 = 0$, therefore $N(T) = \{0\}$. We deduce that $T$ is one-to-one, by Thm. 2.4.

By the Dim. Thm, we deduce that $\text{dim}(R(T)) = \text{dim}(P_2(R)) - \text{dim}(N(T)) = \text{dim}(P_2(R)) = 3$. However, $\text{dim}(P_3(R)) = 4$, therefore $R(T) \neq P_3(R)$ and $T$ is not onto.

From $T\left(a_0 + a_1x + a_2x^2\right) = a_1 + (a_0 + 2a_2)x + a_1x^2 + a_2x^3 = a_1(1 + x^2) + a_0x + a_2(2x + x^3)$ we see that a basis for $R(T)$ is given by $\{1 + x^2, x, 2x + x^3\}$ (clearly we see that it is a generator of $\{T(a_0 + a_1x + a_2x^2)\}$ and is l.i.).

Exercise 14/2.1:

(a) Assume $T$ one-to-one. Let $S = \{v_1, v_2, ..., v_n\} \subset V$ be l.i., and let $S' = \{T(v_1), T(v_2), ..., T(v_n)\}$.
Assume \( a_1 T(v_1) + a_2 T(v_2) + \ldots + a_n T(v_n) = 0_W \) for some scalars \( a_1, \ldots, a_n \in F \). Then, since \( T \) is linear, we have \( T(a_1 v_1 + a_2 v_2 + \ldots + a_n v_n) = 0_W \). But \( T \) is also one-to-one, i.e. \( N(T) = \{ x : T(x) = 0_W \} = \{ 0_V \} \), therefore \( a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0_V \). But \( S = \{ v_1, v_2, \ldots, v_n \} \) is l.i., this implies that \( a_1 = 0, a_2 = 0, \ldots, a_n = 0 \). In conclusion, \( S' = \{ T(v_1), T(v_2), \ldots, T(v_n) \} \) is l.i.

Converse: assume by contradiction that \( T \) is not one-to-one. Then \( N(T) \neq \{ 0_V \} \), therefore there is \( v \in N(T) \) with \( v \neq 0_V \), with \( T(v) = 0_W \). But this is a contradiction, since \( \{ v \} \) is l.i., while \( \{ T(v) = 0_W \} \) is l.d. In conclusion, \( T \) must be one-to-one.

(b) From left to right: directly by (a). From right to left: by Exercise 13 (was proved in class and at Midterm 1).

(c) By property (a), since \( T \) is one-to-one and \( \beta \) is l.i. (it is a basis), we deduce that \( T(\beta) \) is l.i. Now since \( T \) is also onto, and by the Dim. Thm. (also by \( \dim(N(T)) = 0 \) we have that \( \dim(R(T)) = \dim(W) \) and \( \dim(R(T)) = \dim(V) - \dim(N(T)) = n \). Therefore \( \dim(W) = \dim(V) = n \) and \( S(\beta) \) is therefore a basis of \( W \), because \( S(\beta) \) is l.i. and contains exactly \( n \) distinct vectors.

**Exercise 17/2.1:**

(a) By the Dimension Thm., we have \( \dim(N(T)) + \dim(R(T)) = \dim(V) \). If \( \dim(V) < \dim(W) \), then \( \dim(R(T)) \leq \dim(V) < \dim(W) \), therefore \( \dim(R(T)) < \dim(W) \). This shows that \( R(T) \neq W \), i.e. \( T \) is not onto.

(b) We apply again Dimension Thm: \( \dim(N(T)) + \dim(R(T)) = \dim(V) \).

We also know \( \dim(R(T)) \leq \dim(W) \), therefore \( \dim(V) > \dim(W) \geq \dim(R(T)) = \dim(V) - \dim(N(T)) \), i.e. \( 0 > -\dim(N(T)) \) or \( \dim(N(T)) > 0 \). Therefore, \( N(T) \neq \{ 0_V \} \), and by Thm. 2.4, \( T \) is not one-to-one.

Note that these general properties (a) and (b) could have been applied to the linear transformations from Exercises 2 and 5 above.

**Exercise 15/2.2:**

(a) Clearly the zero transformation \( T_0 : V \rightarrow W \) belongs to \( S^0 \), because \( T_0(x) = 0_W \) for any \( x \in V \), including any \( x \in S \).

If \( T_1, T_2 \in S^0 \), and if \( x \in S \), then \( (T_1 + T_2)(x) = T_1(x) + T_2(x) = 0_W + 0_W = 0_W \), for any \( x \in S \), therefore \( T_1 + T_2 \in S^0 \).

Similarly, if \( T \in S^0 \) and \( c \in F \), then for any \( x \in S \): \( (cT)(x) = cT(x) = c0_W = 0_W \), therefore \( cT \in S^0 \). In conclusion, \( S^0 \) is a subspace.

(b) If \( T \in S^0_2 \), then \( T(x) = 0 \) for any \( x \in S_2 \); but \( S_1 \subset S_2 \), therefore \( T(x) = 0 \) for any \( x \in S_1 \), i.e. \( T \in S^0_1 \).
(c) If \( T \in V_1^0 \cap V_2^0 \), then \( T \in V_1^0 \) and \( T \in V_2^0 \). Therefore \( T(x) = 0 \) for any \( x \in V_1 \) and \( T(x) = 0 \) for any \( x \in V_2 \). This implies that for any \( x = u + v \in V_1 + V_2 \) with \( u \in V_1 \) and \( v \in V_2 \), then \( T(x) = T(u + v) = T(u) + T(v) = 0 + 0 = 0 \), therefore \( T \in (V_1 + V_2)^0 \). In other words, \( V_1^0 \cap V_2^0 \subset (V_1 + V_2)^0 \).

To show the other inclusion: we have \( V_1 \subset V_1 + V_2 \) (since \( V_2 \) is a subspace and \( 0 \in V_2 \)), therefore from (b), \( (V_1 + V_2)^0 \subset V_1^0 \). Similarly, \( V_2 \subset V_1 + V_2 \) (since \( V_1 \) is a subspace and \( 0 \in V_1 \)), therefore again from (b), \( (V_1 + V_2)^0 \subset V_2^0 \). These last two statements imply \( (V_1 + V_2)^0 \subset V_1^0 \cap V_2^0 \).

In conclusion: \( (V_1 + V_2)^0 = V_1^0 \cap V_2^0 \).

**Exercise 3/2.4:** By Thm. 2.19:
(a) not isomorphic, the dimensions are different: 3 and 4.
(b) isomorphic, the dimensions are the same = 4.
(c) isomorphic, the dimensions are the same = 4.
(d) not isomorphic, the dimensions are different: \( \dim(V) = 3 \) and \( \dim W = 4 \).

**Exercise 14/2.4:** Let \( T(\begin{pmatrix} a & a + b \\ 0 & c \end{pmatrix}) = (a, b, c) \). Finalize the problem by showing that \( T \) is linear, one-to-one and onto.