Midterm Solutions

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1. In $M_{2\times 3}(\mathbb{F})$, prove that the set

$$\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$$

is linearly dependent.

Solution:

$$(1)\cdot \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + (1)\cdot \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} + (1)\cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} + (-1)\cdot \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} + (-1)\cdot \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = 0$$

clearly proves linear dependence.

2. Let $T : V^n \to W^m$ be a linear transformation from an n-dimensional vector space V to an m-dimensional vector space W.
(i) Prove that if $n > m$, then T cannot be injective.
(ii) Prove that if $n < m$, then T cannot be surjective.

Solution: This was a homework problem. I will approach this problem differently from the way I did in the homework.

(i) Suppose that $n > m$. Then by the Dimension Formula,

$$\text{nullity}(T) + \text{rank}(T) = n > m.$$

Subtracting $\text{rank}(T)$ on both sides we get

$$\text{nullity}(T) > m - \text{rank}(T).$$

Since $R(T)$ is always a subspace of W, it follows that $\text{rank}(T) \leq \dim(W) = m$. Therefore $\text{nullity}(T) > 0$ and hence $N(T)$ is a nontrivial subspace of V, i.e. $N(T) \neq \{0\}$. This completes the proof.

(ii) Suppose that $n < m$. Then by the Dimension Formula,

$$\text{nullity}(T) + \text{rank}(T) = n < m.$$

Subtracting rank(T) on both sides we get

$$\text{nullity}(T) < m - \text{rank}(T).$$

Since nullity$(T) \geq 0$, it follows that $m-$rank$(T) > 0$, or $m >$ rank(T). As $R(T)$ is a subspace of W, by considering their respective dimensions, we get that $R(T) \subsetneq W$. This completes the proof.

3. Let V and W be finite dimensional vector spaces with ordered basis $\beta = \{v_1, ..., v_n\}$ and $\gamma = \{w_1, ..., w_m\}$ respectively. Define the linear transformation $T_{ij} : V \to W$ such that $T_{ij}(v_k) = \delta_{kj} w_i$ where δ_{kj} is the Kronecker delta-function. Prove that $\{T_{ij} : 1 \leq i \leq m \;\; 1 \leq j \leq n\}$ is a basis of $\mathcal{L}(V, W)$.

Solution: Since we know that $\dim(\mathcal{L}(V, W)) = mn$, it suffices to prove that the set is linearly independent (since there are precisely mn elements in this set.)

Let $a_1^1, a_1^2, ..., a_1^m, a_2^1, a_2^2, ..., a_2^m, ..., a_n^1, a_n^2, ...a_n^m$ be scalars such that

$$\sum_{i,j} a_j^i T_{ij} = 0 = T_0$$

where this sum is over all appropriate i's and j's. Let us evaluate both sides with the vector v_k, where k is some integer such that $1 \leq k \leq n$. Then the LHS equals

$$\sum_{i,j} a_j^i T_{ij}(v_k) = \sum_{i,j} a_j^i \delta_{kj} w_i \stackrel{*}{=} \sum_i a_k^i w_i,$$

where $*$ follows from the fact that the middle sum over j's are all zeros except precisely when $j = k$. This is because of the Kronecker delta function. Now evaluating v_k on the RHS gives us 0. Therefore

$$\sum_i a_k^i w_i = 0.$$

But since γ is a linearly independent set, it follows that $a_k^1 = a_k^2 = \cdots = a_k^m = 0$. Now since k was arbitrary, we conclude $a_j^i = 0$ for all i's and j's. This completes our proof.

4. let V be a vector space, $T : V \to V$ be a linear transformation. Prove that $T^2 = 0$ if and only if $R(T) \subseteq N(T)$.

Solution: $T^2 = 0$ if and only if for all $v \in V$, $0 = T^2(v) = T(T(v))$ if and only if for all $v \in V$, $T(v) \in N(T)$ if and only if $R(T) \subseteq N(T)$. This completes our proof.

5. For any finite dimensional vector space V of dimension n with an ordered basis β, show that the coordinate map $\phi_\beta : V \to \mathbb{R}^n$ defined by $\phi_\beta(x) = [x]_\beta$ is a linear transformation which is both one-to-one and onto.

Solution:

Linearity: Let $x, y \in V$ and c be any scalar. Then we can write $x = \sum_i a_i\beta_i$ and $y = \sum_l b_l\beta_l$ where a_i, b_l are scalars and $\beta = \{\beta_1, ..., \beta_n\}$. Then

$$\begin{aligned}
\phi_\beta(cx+y) &= \phi_\beta\left(\sum_i (ca_i + b_i)\beta_i\right) \\
&= \begin{pmatrix} ca_1 + b_1 \\ \vdots \\ ca_n + b_n \end{pmatrix} \\
&= c\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \\
&= c\phi_\beta(x) + \phi_\beta(y).
\end{aligned}$$

Therefore ϕ_β is linear.

One-to-one: Let $x \in V$ be such that $\phi_\beta(x) = 0$. Furthermore, let us write $x = \sum_i a_i\beta_i$. We need to prove that $x = 0$. So

$$\phi_\beta(x) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix},$$

and hence $a_i = 0$ for all i.

Onto: is immediate. Let

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

be any arbitrary vector in $\mathbb{R}^n$. Then define $x = \sum_i a_i\beta_i$. Then clearly

$$\phi_\beta(x) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.$$

This completes our proof.