Math 115a: Selected Solutions for HW 7

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Exercise 5.2.2b: For each of the following matrices $A \in M_n(\mathbb{R})$, test A for diagonalizability, and if A is diagonalizable, find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

Solution: The characteristic polynomial is

$$\begin{aligned} p(t) &= (1-t)^2 - 9 \\ &= t^2 - 2t - 8 \\ &= (t-4)(t+2). \end{aligned}$$

Since the dimension of our vector space is 2 and we have found 2 distinct eigenvalues: 4 and -2, we conclude that A is diagonalizable. Furthermore,

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Exercise 5.2.3b: For each of the following linear operators T on a vector space V, test T for diagonalizability, and if T is diagonalizable, find a basis β for V such that $[T]_\beta$ is a diagonal matrix: $V = P_2(\mathbb{R})$ and T is defined by $T(ax^2 + bx + c) = cx^2 + bx + a$.

Solution: By inspection, we see that $T(x^2 + x + 1) = x^2 + x + 1$ and $T(x) = x$. Therefore $Eig_1(T) = \text{span}(\{x^2 + x + 1, x\})$. Secondly, we see that $T(x^2 - 1) = -(x^2 - 1)$ and therefore $Eig_{-1}(T) = \text{span}(\{x^2 - 1\})$. So $Eig_1(T) = Eig_{-1}(T) = 3 = dim(V)$ and hance T is diagonalizable. Define $\beta = \{x^2 + x + 1, x^2 - 1, x\}$. Then

$$[T]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Of course β isn't unique. Any permutation of the ordering will give you a diagonal matrix, perhaps with the diagonal entries permuted. BE CAREFUL: This proof is done in a rather extemporaneous fashion. I would advise you to do it via the standard approach of looking at the characteristic polynomial, eigenvalues, etc.

Exercise 5.2.8: Suppose that $A \in M_n(\mathbb{F})$ has two distinct eigenvalues, λ_1 and λ_2, and that $\dim(E_{\lambda_1}) = n - 1$. Prove that A is diagonalizable.

Solution: Since $\dim(E_{\lambda_2}) \geq 1$, and we know that $\dim(E_{\lambda_1}) + \dim(E_{\lambda_2}) \leq n$, this forces $\dim(E_{\lambda_2}) = 1$. Let E_i be the eigenspace of A corresponding to λ_i. Since $E_1 \cap E_2 = \{0\}$, and under dimension considerations, we conclude that $V = E_1 \oplus E_2$. By Theorem 5.11, we're done. (Note: Make sure you can prove that E_1 and E_2 form a direct sum.)

Exercise 5.2.18a: Prove that if T and U are simultaneously diagonalizable operators, then T and U commute.

Solution: Let β be the basis that simultaneously diagonalizes T and U. Since diagonal matrices commute with each other, we conclude that

$$[TU]_\beta = [T]_\beta[U]_\beta = [U]_\beta[T]_\beta = [UT]_\beta.$$

Since $[TU]_\beta = [UT]_\beta$, we conclude that T commutes with U. (We've only proven that with respect to this particular basis that their matrix representations commute. Yet we are concluding something that is so much stronger. Why can we do this? Make sure you know.)

Exercise 6.1.9 Let $\beta = \{\beta_1, ..., \beta_n\}$ be a basis for a finite-dimensional inner product space. (a) Prove that if $\langle x, z \rangle = 0$ for all $z \in \beta$, then $x = 0$. (b) Prove that if $\langle x, z \rangle = \langle y, z \rangle$ for all $z \in \beta$, then $x = y$.

Solution: (a) Express x with respect to β: $x = a_1\beta_1 + \cdots a_n\beta_n$. Then

$$\begin{aligned}
\langle x|z \rangle &= \langle \sum_{i=1}^{n} a_i\beta_i|z\rangle \\
&= \sum_{i=1}^{n} a_i\langle \beta_i|z\rangle \quad \text{(By property (i.) and (iii.) of the definition of } \langle\cdot|\cdot\rangle\text{)} \\
&= 0 \quad \text{(By assumption).}
\end{aligned}$$

(b) It suffices to prove that $\langle x-y|z\rangle = 0$, because we can simply use property (i.) of the definition of the inner product to get what we want. But if $\langle x-y|z\rangle = 0$, then by part (a) $x - y = 0 \rightarrow x = y$. This completes our proof.

Exercise 6.1.12: Let $\{v_1, v_2, ..., v_k\}$ be an orthogonal set in V, and let $a_1, a_2, ..., a_k$ be scalars. Prove that

$$\|\sum_{i=1}^{k} a_i v_i\| = \sum_{i=1}^{k} |a_i|^2 \|v_i\|^2.$$

Solution:

$$\begin{aligned}
\|\sum_{i=1}^{k} a_i v_i\| &= \langle \sum_{i=1}^{k} a_i v_i \mid \sum_{j=1}^{k} a_j v_j \rangle \\
&= \sum_{i=1}^{k} a_i \langle v_i \mid \sum_{j=1}^{k} a_j v_j \rangle \\
&= \sum_{i=1}^{k} a_i \overline{\langle \sum_{j=1}^{k} a_j v_j \mid v_i \rangle} \\
&= \sum_{i=1}^{k} a_i \sum_{j=1}^{k} \overline{a_j} \overline{\langle v_j \mid v_i \rangle} \\
&= \sum_{i=1}^{k} a_i \sum_{j=1}^{k} \overline{a_j} \langle v_i \mid v_j \rangle \\
&\overset{*}{=} \sum_{l=1}^{k} a_l \overline{a_l} \langle v_l \mid v_l \rangle \\
&= \sum_{l=1}^{k} |a_l|^2 \|v_l\|^2
\end{aligned}$$

where $*$ used the orthogonality assumption.

Exercise 6.1.17: Let T be a linear operator on an inner product space V, and suppose that $\|T(x)\| = \|x\|$ for all x. Prove that T is one-to-one.

Solution: Let $x \in N(T)$. Then $||x|| = ||T(x)|| = ||0||$. By property (iv.) of $\langle \cdot \mid \cdot \rangle$ we conclude $x = 0$. Therefore T is injective. This completes the proof.