Math 115a: Selected Solutions for HW 6 + More

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December 5, 2005

Exercise 5.1.7a: Let T be a linear operator on a finite-dimensional vector space V. We define the **determinant** of T, denoted $\det(T)$, as follows: Choose any ordered basis β for V, and define $\det(T) = \det([T]_\beta)$. Prove that the preceding definition is independent of the choice of an ordered basis for V. That is, prove that if β and γ are two ordered bases for V, then $\det([T]_\beta) = \det([T]_\gamma)$.

Solution: Let β and γ be two ordered bases for V. Then we have the following equality:

$$[T]_\beta = Q[T]_\gamma Q^{-1},$$

where Q is the change of coordinate matrix from β to γ. Taking determinants on both sides and noting the commuting property of determinants, our claim is immediate.

Exercise 5.1.8a: Prove that a linear operator T on a finite-dimensional vector space is invertible if and only if zero is not an eigenvalue of T.

Solution: On a finite dimensional vector space V, T is invertible if and only if T is injective, i.e. $N(T) = \{0\}$. This means that the only vector v such that $T(v) = 0$ is the zero vector. But these vectors are precisely the eigenvectors of T corresponding to eigenvalue 0. Therefore we see that there must be no such eigenvectors, as eigenvectors must be nonzero. The reverse direction is basically the same argument, only ran backwards.

Exercise 5.1.9: Prove that the eigenvalues of an upper triangular matrix M are the diagonal entries of M.

Solution: Let $M \in M_n(\mathbb{F})$ be upper triangular, ie-

$$M = (m_{ij}),$$

where $m_{ij} = 0$ whenever $i > j$. Since the eigenvalues of M are precisely the roots of its characteristic polynomial, the roots of

$$p(t) = \det(M - tI_n) \stackrel{*}{=} \prod_{i=1}^{n} (m_{ii} - t)$$

are precisely the eigenvalues. ($*$ is from the fact that $M - tI_n$ is also an upper triangular matrix and the determinant of an upper triangular matrix is just the product of its diagonal entries.) Therefore the eigenvalues of M are m_{ii}, $1 \leq i \leq n$. This completes the proof.

Exercise 5.1.11: A **scalar matrix** is a square matrix of the form λI for some scalar λ; that is, a scalar matrix is a diagonal matrix in which all the diagonal entries are equal. (a) Prove that if a square matrix A is similar to a scalar matrix λI, then $A = \lambda I$. (b) Show that a diagonalizable matrix having only one eigenvalue is a scalar matrix.

Solution: (a) Let $A \in M_n(\mathbb{F})$ be such that $A = QSQ^{-1}$ for some invertible Q and scalar matrix S. More to the point, let $S = \lambda I$ for some $\lambda \in \mathbb{F}$. Then

$$A = Q(\lambda I)Q^{-1} = \lambda QQ^{-1}I = \lambda I.$$

This completes the proof.

(b) Let A be diagonalizable having only one eigenvalues, say, λ. Then

$$A = QDQ^{-1}$$

for some invertible Q and diagonal D. On the other hand, we know that in such a situation, the diagonal entries of D are precisely the eigenvalues of A. In this case since we only have one eigenvalue, D must be the scalar matrix λI. Then by part (a), $A = \lambda I$. This completes our proof.

Exercise 5.1.14: For any square matrix A, prove that A and A^t have the same characteristic polynomial (and hence the same eigenvalues).

Solution: Let $f(t)$ and $g(t)$ denote the characteristic polynomials of A and A^t, respectively. Then

$$f(t) = \det(A - tI) \stackrel{*}{=} \det((A - tI)^t) = \det(A^t - tI) = g(t),$$

where $*$ should be obvious (as in you should know how to prove this fact). Since $f(t) = g(t)$, their roots coincide. Therefore A and A^t have the same eigenvalues. This completes the proof.

Exercise 5.1.15a: Let T be a linear operator on a vector space V, and let x be an eigenvector of T corresponding to the eigenvalue λ. For any positive integer m, prove that x is an eigenvector of T^m corresponding to the eigenvalue λ^m.

Solution: We prove by induction on m. Base case: $m = 1$ holds, by definition that x is an eigenvector of T corresponding to λ. Inductive hypothesis: Let us suppose that the claim is true for $m > 1$, that is: $T^m(x) = \lambda^m x$. Then

$$T^{m+1}(x) = T(T^m(x)) \stackrel{*}{=} T(\lambda^m(x)) = \lambda^m T(x) = \lambda^m \lambda x = \lambda^{m+1}x,$$

where $*$ is the inductive hypothesis. Therefore the claim is true for $m + 1$. By induction the proof is complete.

Exercise 5.1.17a: Let T be a linear operator on $M_n(\mathbb{R})$ defined by $T(A) = A^t$. Show that ± 1 are the only eigenvalues of T.

Solution: Let λ be an eigenvalue of T and A be an eigenvector of T corresponding to λ. Then

$$\lambda A = T(A) = A^t.$$

Applying T again to everything gives us:

$$\lambda^2 A = T(\lambda A) = T^2(A) = (A^t)^t = A.$$

Therefore $\lambda^2 = 1$ and hence $\lambda = \pm 1$.