# Math 115a: Selected Solutions to HW 5

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**Exercise 2.4.4**: Let $A$ and $B$ be $n \times n$ invertible matrices. Prove that $AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

*Solution*: Let $A$ and $B$ be invertible $n \times n$ matrices. Then

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = A(IA^{-1}) = AA^{-1} = I.$$

**Exercise 2.4.5:** Let $A$ be invertible. Prove that $A^t$ is invertible and $(A^t)^{-1} = (A^{-1})^t$.

*Solution:* In proving that $(A^t)^{-1} = (A^{-1})^t$, all we need to do is to verify that it is both a left inverse as well as a right inverse:

$$(A^t)(A^{-1})^t \stackrel{*}{=} (A^{-1}A)^t = (I_n)^t = I_n,$$

where $*$ is a standard property of the transpose operator. Similarly,

$$(A^{-1})^t(A^t) = (AA^{-1})^t = (I_n)^t = I_n.$$

Therefore $A^t$ is invertible and we have verified what its inverse is.

**Exercise 2.4.10**: Let $A$ and $B$ be $n \times n$ matrices such that $AB = I_n$.
(a) Use Exercise 9 to conclude that $A$ and $B$ are invertible.
(b) Prove $A = B^{-1}$ (and hence $B = A^{-1}$).
(c) State and prove analogous results for linear transformations defined on finite-dimensional vector spaces.

*Solution*:
(a) By Exercise 9, if $AB$ is invertible, then so are $A$ and $B$. Clearly $AB = I_n$ is invertible. Therefore our conclusion follows immediately.
(b) We need to show that $A = B^{-1}$, which means that $AB = BA = I_n$. $AB = I_n$ is given to us by assumption, so it suffices to show $BA = I_n$: Multiplying $A$ on the right of $I_n = AB$, we get

$$A = I_nA = ABA.$$

Since $A$ is invertible, there exists $A^{-1}$. Therefore we can multiply on the left by $A^{-1}$ to get

$$A^{-1}A = A^{-1}(ABA) = (A^{-1}A)BA = I_n BA = BA.$$

Reducing $A^{-1}A = I_n$, and we get our conclusion.
(c) Claim: Let $V$ be a $n$-dimensional vector space over $\mathbb{F}$.If $S, T$ are linear operators on $V$ such that $ST : V \to V$ is an isomorphism, then both $S$ and $T$ are isomorphisms.

Proof: Suppose $S, T$ are linear operators on $V$ such that $ST$ is an isomorphism. Let $\beta = \{\beta_1, \beta_2, ..., \beta_n\}$ be an ordered basis for $V$. Let $A$ and $B$ be the matrix representation of $S$ and $T$, respectively, using $\beta$:

$$A = [S]_\beta, \quad B = [T]_\beta.$$

Then $[ST]_\beta = AB$. Since $ST$ is an isomorphism, $AB$ is an invertible matrix. By part (a), both $A$ and $B$ are invertible. Finally, this implies that both $S$ and $T$ are isomorphisms; this completes our proof. **Exercise 2.4.17**: Let $V$ and $W$ be finite-dimensional vector spaces and $T : V \to W$ be an isomorphism. Let $V_0$ be a subspace of $V$.

(a) Prove that $T(V_0)$ is a subspace of $W$.
(b) Prove that dim $(V_0) = \dim(T(V_0))$.

*Solution*:

(a) Let $c \in F$ and $w_1, w_2 \in T(V_0)$. We need to show that $cw_1 + w_2 \in T(V_0)$. Let $v_1, v_2 \in V_0$ such that $T(v_i) = w_i$. Since $V_0$ is a subspace of $V$, it $cv_1 + v_2 \in V_0$. Therefore $T(cv_1 + v_2) = cT(v_1) + T(v_2) = cw_1 + w_2 \in T(V_0)$. Finally, since $T(0) = 0 \in T(V_0)$ we are done.

(b) Let us define $T\big|_{V_0} : V_0 \to T(V_0)$ to be simply the restriction of $T$ to the domain $V_0$. Since $T$ is a isomorphism, $N(T) = \{0\}$. Therefore it follows that $N(T\big|_{V_0}) = \{0\}$. And hence $T\big|_{V_0}$ is injective. By the dimension formula, dim $(V_0)$ = nullity$(T\big|_{V_0})$ + rank$(T\big|_{V_0})$ = rank$(T\big|_{V_0})$ = dim (R$(T\big|_{V_0})$) = dim $(T(V_0))$.

**Exercise 2.4.20**: Let $T : V \to W$ be a linear transformation from an $n$-dimensional vector space $V$ to an $m$-dimensional vector space $W$. Let $\beta$ and $\gamma$ be ordered bases for $V$ and $W$, respectively. Prove that rank$(T)$ = rank$(L_A)$ and that nullity$(T)$ = nullity$(L_A)$, where $A = [T]_\beta^\gamma$.

We begin with the following claim: If $S : V^m \to W^m$ is an isomorphism and $T : W^m \to Z^n$ is a linear transformation, then rank$(TS)$ = rank$(T)$ and nullity$(TS)$ = nullity $(T)$ (Note: the superscripts on the vector spaces denote dimension).

*Sketch of the proof of the claim*: Let $\{z_1, ..., z_k\}$ be a basis of $R(T)$. Then there exists $\{w_1, ..., w_k\} \subseteq W$ such that $T(w_i) = z_i$. Let $\{w_{k+1}, ..., w_m\}$ be a basis of $N(T)$. Then it follows that $\{w_1, ..., w_m\}$ is a basis for $W$ (prove this). Since $S$ is an isomorphism between $V$ and $W$, it is injective and surjective. Therefore there exists $\{v_1, ..., v_m\} \subseteq V$ such that $S(v_i) = w_i$. Furthermore, $\{v_{k+1}, ..., v_m\}$ is a basis for $N(TS)$ (why?) Therefore the $R(TS) = \text{span}\{TS(v_1), ..., TS(v_k)\}$ $= \text{span}\{T(w_1), ..., T(w_k)\} = \text{span } \{z_1, ..., z_k\} = R(T)$, and all of these sets of vectors for bases for their respective vector spaces. Apply dimension to both sides and we get $\text{rank}(TS) = \text{rank}(T)$. Now use the dimension formula to get $\text{nullity}(TS) = \text{nullity } (T)$.

A similar claim goes as follows: Let $T : V^m \to W^n$ be a linear transformation and $S : W^n \to Z^n$ be an isomorphism. Then $\text{nullity}(ST) = \text{nullity } (T)$ and $\text{rank}(ST) = \text{rank}(T)$.

The proof of this claim is almost the same as the proof of the first claim.

*Proof of the Exercise*: Using Figure 2.2 we see that $L_A = \phi_\gamma \circ T \circ \phi_\beta^{-1}$. Since $\phi_\gamma$ is an isomorphism, by the first claim, rank $(\phi_\gamma \circ T \circ \phi_\beta^{-1}) =$ rank $(T \circ \phi_\beta^{-1})$. Since $\phi_\beta^{-1}$ is an isomorphism, by the second claim, rank $(T \circ \phi_\beta^{-1}) =$ rank $(T)$. Putting all of this together we get the following string of equalities:

$$\text{rank}(L_A) = \text{rank}(\phi_\gamma \circ T \circ \phi_\beta^{-1}) = \text{rank}(T).$$

Proving equality of the nullity is literally the same as that for the rank; simply replace all occurrences of "rank" with "nullity".

**Exercise 2.5.10**: Prove that if $A$ and $B$ are similar $n \times n$ matrices, then $\text{tr}(A) = \text{tr}(B)$.

*Solution:* Let $Q$ be the matrix such that $A = QBQ^{-1}$. Then

$$tr(A) = tr(QBQ^{-1}) \stackrel{*}{=} tr(QQ^{-1}B) = tr(B),$$

where $*$ is by Exercise 13 of Section 2.3.

**Exercise 2.5.13**: Let $V$ be a finite dimensional vector space over a field $F$, and let $\beta = \{x_1, ...x_n\}$ be an ordered basis for $V$. Let $Q$ be an $n \times n$ invertible matrix with entries from $F$. Define

$$x'_j = \sum_{i=1}^{n} Q_{ij} x_i \quad \text{for } 1 \leq j \leq n,$$

and set $\beta' = \{x'_1, ...x'_n\}$. Prove that $\beta'$ is a basis for $V$ and hence that $Q$ is the change of coordinate matrix changing $\beta'$-coordinates into $\beta$-coordinates.

*Solution*: Let us define the following matrices:

$$A = \begin{pmatrix} | & | & | & | \\ x_1 & x_2 & \dots & x_n \\ | & | & | & | \end{pmatrix} \; A' = \begin{pmatrix} | & | & | & | \\ x'_1 & x'_2 & \dots & x'_n \\ | & | & | & | \end{pmatrix},$$

where the $x_i$'s and $x'_j$'s are columns of $A$ and $A'$, respectively. Then our assumption can be translated into the language of matrix multiplication:

$$A' = QA.$$

We notice that since the columns of $A$ make up a basis (hence are linearly independent) we see that $A$ is invertible. Therefore $A'$ is also an invertible matrix, since it's the product of two invertible matrices. Therefore the columns of $A'$ are linearly independent, implying that $\beta'$ makes up a basis for $V$ (since there are $n$ vectors). By construction, $Q$ is the change of coordinates matrix.