Math 115a: Selected Solutions for HW 3

Paul Young

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Exercise 2.1.3: Prove that T is a linear transformation, and find bases for both $N(T)$ and $R(T)$. Then compute the nullity and rank of T, and verify the dimension theorem. Finally, use the appropriate theorems in this section to determine whether T is one-to-one or onto: Define $T : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$$

Solution: We first prove that T is a linear transformation. Let $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$ and let $c \in \mathbb{R}$.

$$\begin{aligned} T(cx + y) &= T(c(x_1, x_2) + (y_1, y_2)) \\ &= T(cx_1 + y_1, cx_2 + y_2) \\ &= ((cx_1 + y_1) + (cx_2 + y_2), 0, (2(cx_1 + y_1) - (cx_2 + y_2)) \\ &= c(x_1 + x_2, 0, 2x_1 - x_2) + (y_1 + y_2, 0, 2y_1 - y_2) \\ &= cT(x_1, x_2) + T(y_1, y_2), \end{aligned}$$

and hence T is linear. Next we figure out what $N(T)$ and $R(T)$ look like. Let $x = (x_1, x_2) \in N(T)$. Then

$$\begin{aligned} 0 &= T(x_1, x_2) = (x_1 + x_2, 0, 2x_1 - x_2) \to \\ & \quad x_1 + x_2 = 0, \quad 2x_1 - x_2 = 0 \to \\ & \quad x_1 = 0, \quad x_2 = 0. \end{aligned}$$

Therefore we conclude that $N(T) = \{0\}$, so that the basis for $N(T)$ would be $\{0\}$. We now look at the image space. Generally, what we do is take a basis of the domain, and then transform each of these basis elements by T to see what we get. More specifically, let $\beta =$ be the canonical basis for $\mathbb{R}^2$– that is, $\beta = \{(1, 0), (0, 1)\}$. Then

$$\begin{aligned} T(1, 0) &= (1, 0, 2) \\ T(0, 1) &= (1, 0, -1) \end{aligned}$$

and hence $R(T) = \text{span}(\{(1, 0, 2), (1, 0, -1)\})$. Since these two vectors are linearly independent, we conclude that this is actually a basis for $R(T)$. Therefore after computing $N(T)$ and $R(T)$, we conclude that nullity$(T) = \dim(N(T)) = 0$ and rank$(T) = \dim(R(T)) = 2$. This is clearly consistent with the dimension formula:

$$\begin{array}{rcl} \dim(\mathbb{R}^2) & = & \text{nullity}(T) + \text{rank}(T) \\ 2 & = & 0 \quad + \quad 2. \end{array}$$

Lastly, since $N(T) = \{0\}$, by theorem 2.4 T is injective. Now since the range space is $\mathbb{R}^3$, which is larger in dimension than that of the domain space, we conclude that T cannot be onto.

Exercise 2.1.9ace: In this exercise, $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a function. For each of the following parts, state why T is *not* linear.
(a) $T(a_1, a_2) = (1, a_2)$
(c) $T(a_1, a_2) = (\sin a_1, 0)$
(e) $T(a_1, a_2) = (a_1 + 1, a_2)$

Solution: (a) Fails under scalar multiplication when the scalar is 0. (c) Fail addition for practically every pair of vectors. (e) Fails under scalar multiplication when the scalar is 2. (Note: There may be other reasons why these functions fail to be linear. I only picked out the most obvious reasons.)

Exercise 2.1.14c: Let V and W be vector spaces and $T : V \to W$ be linear. Suppose $\beta = \{v_1, v_2, ..., v_n\}$ is a basis for V and T is one-to-one and onto. Prove that $T(\beta) = \{T(v_1), T(v_2), ..., T(v_n)\}$ is a basis for W.

Solution: In order to prove that $T(\beta)$ is a basis, we need to show two things: $T(\beta)$ is a linearly independent set and span $(T(\beta)) = W$.

<u>L.I.</u>: Let $a_1, a_2, ..., a_n \in \mathbb{F}$ be scalars such that

$$\sum_{i=1}^{n} a_i T(v_i) = 0.$$

By linearity of T, we may rewrite the left hand side as:

$$\sum_{i=1}^{n} a_i T(v_i) \quad = \quad T\left(\sum_{i=1}^{n} a_i v_i\right)$$

Since T is injective, $N(T) = \{0\}$. Therefore

$$\sum_{i=1}^{n} a_i v_i \in N(T) \to \sum_{i=1}^{n} a_i v_i = \overrightarrow{0}.$$

Since $\{v_1, v_2, ..., v_n\}$ is a basis for V, it is a linearly independent set. Therefore the last equality we got implies that $a_i = 0$ for all i. Therefore we've proven

L.I. for $T(\beta)$.

span $(T(\beta)) = W$: Let $w \in W$ be arbitrary. Since T is surjective, there exists $v \in V$ such that $T(v) = w$. We express this v in terms of the basis β: $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$, where the c_i's are scalars. Therefore we can write the following expressions:

$$\begin{aligned} w &= T(v) = T\left(\sum_{i=1}^{n} c_iv_i\right) \\ &\stackrel{\star}{=} \sum_{i=1}^{n} c_iT(v_i) \end{aligned}$$

where $\stackrel{\star}{=}$ is given to us by linearity of T. Looking at what we've just done, we have written w as a linear combination of elements from $T(\beta)$. Therefore $w \in$ span$(T(\beta))$. Since w was arbitrarily chosen, $W \subseteq$ span$(T(\beta))$. We note that this is enough to establish equality because span$(T(\beta)) \subseteq R(T) \subseteq W$ is given to us for free. Therefore span $(T(\beta)$.

This completes our proof.

Exercise 2.1.17: Let V and W be finite-dimensional vector spaces and $T : V \to W$ be linear.
(a) Prove that if $\dim(V) < \dim(W)$, then T cannot be onto.
(b) Prove that if $\dim(V) > \dim(W)$, then T cannot be one-to-one.

Solution:

(a) Suppose for the sake of contradiction that T is onto. Then $\text{rank}(T) = \dim(W)$. We are given the following chain of relations:

$$\begin{aligned} \dim(W) &> \dim(V) \stackrel{*}{=} \text{nullity}(T) + \text{rank}(T) \\ &= \dim(V) = \text{nullity}(T) + \dim(W) \to \\ \dim(W) &> \text{nullity}(T) + \dim(W) \end{aligned}$$

where $\stackrel{*}{=}$ is given to us by the Dimension Formula. But this means that nullity(T) must be a negative number, which is nonsense. This is our contradiction and therefore T cannot be onto.

(b) Let us suppose–for the sake of contradiction–that T is injective. Then by Theorem 2.4, $\text{nullity}(T) = \dim(N(T)) = 0$. Then we are given the chain of relations:

$$\begin{aligned} \dim(W) &< \dim(V) \stackrel{*}{=} \text{nullity}(T) + \text{rank}(T) \\ &= 0 + \text{rank}(T) \to \\ \dim(W) &< \text{rank}(T). \end{aligned}$$

But this is clearly impossible because $R(T)$ is a subspace of W and therefore always has dimension less than or equal to the dimension of W. This is our contradiction; therefore T is never injective.

Exercise 2.1.18: Give an example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $N(T) = R(T)$.

Solution: Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$T(a_1, a_2) = (0, a_1).$$

I leave it to the students to verify that $N(T) = \text{span}(\{(0,1)\}) = R(T)$

Exercise 2.1.24: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$. Include figures for each of the following parts.
(a) Find a formula for $T(a, b)$, where T represents the projection on the y-axis along the x-axis.
(b) Find a formula for $T(a, b)$, where T represents the projection on the y-axis along the line $L = \{(s, s) : s \in \mathbb{R}\}$.

Solution: I leave it to the students to draw the figures. I'll only construct the formulas.

(a) Let $v = (v_1, v_2) \in \mathbb{R}^2$. Since $\mathbb{R}^2 = W_1 \oplus W_2$ where $W_1 = \text{span}(\{(1,0)\})$, $W_2 = \text{span}(\{(0,1)\})$, we will write v with respect to this decomposition: $v = v_x + v_y$, such that $v_x = (v_1, 0)$, $v_y = (0, v_2)$. Then by the definition of projection on the y-axis along the x-axis,

$$T(v) = v_y = (0, v_2)$$

(b) Let $v = (v_1, v_2) \in \mathbb{R}^2$. Since $\mathbb{R}^2 = W_1 \oplus W_2$ where $W_1 = \text{span}(\{(1,1)\})$, $W_2 = \text{span}(\{(0,1)\})$, we will write v with respect to this decomposition: $v = v_L + v_y$, such that $v_L = (v_1, v_1)$, $v_y = (0, v_2 - v_1)$. Then by the definition of projection on the y-axis along L,

$$T(v) = v_y = (0, v_2 - v_1).$$