Exercise 2.1.3: Prove that $T$ is a linear transformation, and find bases for both $N(T)$ and $R(T)$. Then compute the nullity and rank of $T$, and verify the dimension theorem. Finally, use the appropriate theorems in this section to determine whether $T$ is one-to-one or onto: Define $T : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$$

Solution: We first prove that $T$ is a linear transformation. Let $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$ and let $c \in \mathbb{R}$.

$$T(cx + y) = T(cx_1 + y_1, cx_2 + y_2) = ((cx_1 + y_1) + (cx_2 + y_2), 0, (2(cx_1 + y_1) - (cx_2 + y_2)) = c(x_1 + x_2, 0, 2x_1 - x_2) + (y_1 + y_2, 0, 2y_1 - y_2) = cT(x_1, x_2) + T(y_1, y_2),$$

and hence $T$ is linear. Next we figure out what $N(T)$ and $R(T)$ look like. Let $x = (x_1, x_2) \in N(T)$. Then

$$0 = T(x_1, x_2) = (x_1 + x_2, 0, 2x_1 - x_2) \Rightarrow$$

$$x_1 + x_2 = 0, \quad 2x_1 - x_2 = 0 \Rightarrow$$

$$x_1 = 0, \quad x_2 = 0.$$

Therefore we conclude that $N(T) = \{0\}$, so that the basis for $N(T)$ would be $\{0\}$. We now look at the image space. Generally, what we do is take a basis of the domain, and then transform each of these basis elements by $T$ to see what we get. More specifically, let $\beta$ be the canonical basis for $\mathbb{R}^2$—that is, $\beta = \{(1, 0), (0, 1)\}$. Then

$$T(1, 0) = (1, 0, 2)$$
$$T(0, 1) = (1, 0, -1)$$

and hence $R(T) = \text{span}\{((1, 0, 2), (1, 0, -1))\}$. Since these two vectors are linearly independent, we conclude that this is actually a basis for $R(T)$. Therefore
after computing $N(T)$ and $R(T)$, we conclude that nullity($T$) = dim($N(T)$) = 0 and rank($T$) = dim($R(T)$) = 2. This is clearly consistent with the dimension formula:

$$\dim(\mathbb{R}^2) = \text{nullity}(T) + \text{rank}(T)$$

$$2 = 0 + 2.$$  

Lastly, since $N(T) = \{0\}$, by theorem 2.4 $T$ is injective. Now since the range space is $\mathbb{R}^3$, which is larger in dimension than that of the domain space, we conclude that $T$ cannot be onto.

**Exercise 2.1.9ace:** In this exercise, $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a function. For each of the following parts, state why $T$ is not linear.

(a) $T(a_1, a_2) = (1, a_2)$
(c) $T(a_1, a_2) = (\sin a_1, 0)$
(e) $T(a_1, a_2) = (a_1 + 1, a_2)$

**Solution:** (a) Fails under scalar multiplication when the scalar is 0. (c) Fail addition for practically every pair of vectors. (e) Fails under scalar multiplication when the scalar is 2. (Note: There may be other reasons why these functions fail to be linear. I only picked out the most obvious reasons.)

**Exercise 2.1.14c:** Let $V$ and $W$ be vector spaces and $T : V \rightarrow W$ be linear. Suppose $\beta = \{v_1, v_2, ..., v_n\}$ is a basis for $V$ and $T$ is one-to-one and onto. Prove that $T(\beta) = \{T(v_1), T(v_2), ..., T(v_n)\}$ is a basis for $W$.

**Solution:** In order to prove that $T(\beta)$ is a basis, we need to show two things: $T(\beta)$ is a linearly independent set and span ($T(\beta)$) = $W$.

L.I.: Let $a_1, a_2, ..., a_n \in \mathbb{F}$ be scalars such that

$$\sum_{i=1}^{n} a_i T(v_i) = 0.$$  

By linearity of $T$, we may rewrite the left hand side as:

$$\sum_{i=1}^{n} a_i T(v_i) = T\left(\sum_{i=1}^{n} a_i v_i\right)$$  

Since $T$ is injective, $N(T) = \{0\}$. Therefore

$$\sum_{i=1}^{n} a_i v_i \in N(T) \rightarrow \sum_{i=1}^{n} a_i v_i = 0.$$  

Since $\{v_1, v_2, ..., v_n\}$ is a basis for $V$, it is a linearly independent set. Therefore the last equality we got implies that $a_i = 0$ for all $i$. Therefore we’ve proven...
L.I. for $T(\beta)$. 

$\text{span } (T(\beta)) = W$: Let $w \in W$ be arbitrary. Since $T$ is surjective, there exists $v \in V$ such that $T(v) = w$. We express this $v$ in terms of the basis $\beta$: $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$, where the $c_i$’s are scalars. Therefore we can write the following expressions:

$$w = T(v) = T\left( \sum_{i=1}^{n} c_i v_i \right)$$

$$= \sum_{i=1}^{n} c_i T(v_i)$$

where $\ast$ is given to us by linearity of $T$. Looking at what we’ve just done, we have written $w$ as a linear combination of elements from $T(\beta)$. Therefore $w \in \text{span}(T(\beta))$. Since $w$ was arbitrarily chosen, $W \subseteq \text{span}(T(\beta))$. We note that this is enough to establish equality because $\text{span}(T(\beta)) \subseteq R(T) \subseteq W$ is given to us for free. Therefore $\text{span } (T(\beta))$.

This completes our proof.

**Exercise 2.1.17**: Let $V$ and $W$ be finite-dimensional vector spaces and $T : V \to W$ be linear.

(a) Prove that if $\dim(V) < \dim(W)$, then $T$ cannot be onto.

(b) Prove that if $\dim(V) > \dim(W)$, then $T$ cannot be one-to-one.

**Solution:**

(a) Suppose for the sake of contradiction that $T$ is onto. Then $\text{rank}(T) = \dim(W)$. We are given the following chain of relations:

$$\dim(W) > \dim(V) \overset{\ast}{=} \text{nullity}(T) + \text{rank}(T)$$

$$= \dim(V) = \text{nullity}(T) + \dim(W) \rightarrow$$

$$\dim(W) > \text{nullity}(T) + \dim(W)$$

where $\ast$ is given to us by the Dimension Formula. But this means that $\text{nullity}(T)$ must be a negative number, which is nonsense. This is our contradiction and therefore $T$ cannot be onto.

(b) Let us suppose—for the sake of contradiction—that $T$ is injective. Then by Theorem 2.4, $\text{nullity}(T) = \dim(N(T)) = 0$. Then we are given the chain of relations:

$$\dim(W) < \dim(V) \overset{\ast}{=} \text{nullity}(T) + \text{rank}(T)$$

$$= 0 + \text{rank}(T) \rightarrow$$

$$\dim(W) < \text{rank}(T).$$
But this is clearly impossible because $R(T)$ is a subspace of $W$ and therefore always has dimension less than or equal to the dimension of $W$. This is our contradiction; therefore $T$ is never injective.

**Exercise 2.1.18:** Give an example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $N(T) = R(T)$.

**Solution:** Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$T(a_1, a_2) = (0, a_1).$$

I leave it to the students to verify that $N(T) = \text{span} \{(0, 1)\} = R(T)$

**Exercise 2.1.24:** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$. Include figures for each of the following parts.

(a) Find a formula for $T(a, b)$, where $T$ represents the projection on the $y$-axis along the $x$-axis.

(b) Find a formula for $T(a, b)$, where $T$ represents the projection on the $y$-axis along the line $L = \{(s, s) : s \in \mathbb{R}\}$.

**Solution:** I leave it to the students to draw the figures. I’ll only construct the formulas.

(a) Let $v = (v_1, v_2) \in \mathbb{R}^2$. Since $\mathbb{R}^2 = W_1 \oplus W_2$ where $W_1 = \text{span} \{(1, 0)\}$, $W_2 = \text{span} \{(0, 1)\}$, we will write $v$ with respect to this decomposition: $v = v_x + v_y$, such that $v_x = (v_1, 0)$, $v_y = (0, v_2)$. Then by the definition of projection on the $y$-axis along the $x$-axis,

$$T(v) = v_y = (0, v_2)$$

(b) Let $v = (v_1, v_2) \in \mathbb{R}^2$. Since $\mathbb{R}^2 = W_1 \oplus W_2$ where $W_1 = \text{span} \{(1, 1)\}$, $W_2 = \text{span} \{(0, 1)\}$, we will write $v$ with respect to this decomposition: $v = v_L + v_y$, such that $v_L = (v_1, v_1)$, $v_y = (0, v_2 - v_1)$. Then by the definition of projection on the $y$-axis along $L$,

$$T(v) = v_y = (0, v_2 - v_1).$$