HOMEWORK 8 - SOLUTIONS

Problem 7.5.25. Remark that we have the relation \( \sigma \tau \sigma^{-1} = (13) = \tau \sigma^2 \). It follows that every element of \( G \) can be written in the form \( \tau^i \sigma^j \) for some \( 0 \leq i \leq 1 \) and \( 0 \leq j \leq 3 \). Hence \( |G| \leq 8 \). Furthermore, \( G \) contains the subgroup \( \langle \sigma \rangle \) of order 4. Hence \( |G| \) is a multiple of 4 by Lagrange’s theorem. But \( G \neq \langle \sigma \rangle \) so \( |G| \geq 8 \). It follows that \( |G| = 8 \).

Problem 7.5.26. Let \( \sigma \in S_n \setminus \{id\} \) (with \( n > 2 \)). Then we can find \( i \) such that \( \sigma(i) \neq i \). Since \( n > 2 \), we can find \( k \neq i, \sigma(i) \). We have \( (\sigma \circ (ik))(i) = \sigma(k) \neq \sigma(i) = ((ik) \circ \sigma)(i) \). Hence \( \sigma \notin Z(S_n) \). So \( Z(S_n) = \{id\} \).

Problem 7.5.29. There are three possible cases:

- \( \sigma = \tau \), then \( \sigma \tau = id \) is a product of (zero) 3-cycles.
- \( \sigma \) and \( \tau \) are disjoint, then \( \sigma = (ij), \tau = (kl) \) for some \( i, j, k, l \) all distinct. We have \( \sigma \tau = (ikl)(ijl) \), a product of two 3-cycles.
- \( \sigma \) and \( \tau \) are neither equal nor disjoint. Then \( \sigma = (ij) \) and \( \tau = (ik) \) for some \( i, j, k \) all distinct. We have \( \sigma \tau = (ikj) \), a product of one 3-cycle.

In conclusion, a product of two permutations is always a product of 3-cycles.

Problem 7.5.36. If \( i \in \{1, \ldots, k\} \), we have \( \sigma \tau \sigma^{-1}(\sigma(a_i)) = \sigma \tau(a_i) = \sigma(a_{i+1}) \) (indexes being considered modulo \( k \)). If \( x \notin \{\sigma(a_1), \ldots, \sigma(a_k)\} \), then \( \sigma^{-1}(x) \notin \{a_1, \ldots, a_k\} \). It follows that \( \tau \sigma^{-1}(x) = \sigma^{-1}(x) \), and \( \sigma \tau \sigma^{-1}(x) = x \). Hence, \( \sigma \tau \sigma^{-1} = (\sigma(a_1) \ldots \sigma(a_k)) \).

Problem 7.5.40. Note that an element of \( A_n \) is a product of an even number of transpositions. So it suffices to prove that if \( \sigma, \tau \) are transpositions, \( \sigma \tau \) is a product of \( n \)-cycles. For \( n = 3 \), this is what we have done in problem 7.5.19. Assume \( n \geq 4 \). We have the cases:

- \( \sigma = \tau \), then \( \sigma \tau = id \) is a product of (zero) \( n \)-cycles.
- \( \sigma \) and \( \tau \) are disjoint, then \( \sigma = (ij), \tau = (kl) \) for some \( i, j, k, l \) all distinct. Denote by \( a_1, \ldots, a_r \) the remaining elements of \( \{1, \ldots, n\} \). We have \( \sigma \tau = (ia_r \ldots a_k)(ikj)(a_1 \ldots a_r) \).
- \( \sigma \) and \( \tau \) are neither equal nor disjoint. Then \( \sigma = (ij) \) and \( \tau = (ik) \) for some \( i, j, k \) all distinct. Denote by \( a_1, \ldots, a_r \) the remaining elements of \( \{1, \ldots, n\} \). We have \( \sigma \tau = (ikj) = (ia_r \ldots a_1 k)(ikj)(a_1 \ldots a_r) \).

So every element of \( A_n \) is a product of \( n \)-cycles.
Problem 7.5.42. We know that the transpositions generate $S_n$.

- Step 1: we prove that the transpositions $s_i = (ii + 1)$ for $i = 1, \ldots, n - 1$ generate $S_n$. Proof: for any $i < j$, consider $\sigma = s_{j-2} \cdots s_{i+1}s_i$. Then we have $\sigma(i) = j - 1$ and $\sigma(j) = j$. Hence $\sigma^{-1}s_{j-1}\sigma = (ij)$. It follows that the subgroup generated by the $s_i$ contains all the transpositions, hence is $S_n$.

- Step 2: we prove that the subgroup generated by $(12)$ and $(1 \ldots n)$ generates $S_n$. Indeed, we have $(1 \ldots n)^{i-1}(12)(1 \ldots n)^{-i+1} = s_i$ for all $i$. The claim follows now from the first point.

In conclusion $(12)$ and $(1 \ldots n)$ generate $S_n$.

Problem 7.5.43. Note that by the previous problem, an automorphism $f$ of $S_3$ is entirely determined by its values on $(12)$ and $(132)$. Furthermore, we know that $f(12)$ is an order 2 element, so $(12), (23)$ or $(13)$, and $f(132)$ is an order 3 element, so $(132)$ or $(123)$. This gives 6 cases:

- if $f(12) = (12)$ and $f(132) = (132)$ then $f$ is the identity which is the conjugation by the identity element of $S_3$.
- if $f(12) = (12)$ and $f(132) = (123)$ then $f$ is the conjugation by $(12)$.
- if $f(12) = (23)$ and $f(132) = (132)$ then $f$ is the conjugation by $(132)$.
- if $f(12) = (13)$ and $f(132) = (132)$ then $f$ is the conjugation by $(123)$.

Hence every automorphism of $S_3$ is inner.

Problem 8.5.2. (a) $S_2$ has two elements: the identity, which has signature 1, and the permutation $(12)$ which has signature $-1$. Hence $A_2 = \{\text{id}\}$.

(b) We know that $A_3$ has cardinal $3!/2 = 3$. Since 3 is prime, $A_3$ is cyclic and simple.

Problem 8.5.4. The center $Z$ of $A_n$ is a normal subgroup of $A_n$. But $A_n$ is simple when $n \geq 5$ by theorem 8.26, so $Z = \{\text{id}\}$ or $Z = A_n$. Since $A_n$ is not abelian, $Z \neq A_n$. Hence $Z = \{\text{id}\}$.

Problem 8.5.6. Remark that $|A_5| = 60$. So a subgroup of order 30 in $A_5$ would have index 2, and so would be normal, contradicting theorem 8.26. Thus there exist no such subgroup.

Problem 8.5.8. Assume we have a normal subgroup $N$ of index 2 in $S_n$, for $n \geq 3$, and let $\sigma$ be the (order 2) non identity element in $N$. Since $A_n$ is simple, $N$ is not contained in $A_n$, so $\sigma \notin A_n$ and $A_n \cap N = \{\text{id}\}$. Furthermore $A_nN = S_n$. Indeed, if $\tau \in S_n$, then either
τ ∈ Aₙ, or τ ∉ Aₙ, but then τ = (τσ)σ with τσ ∈ Aₙ.

So we have Aₙ, N normal, Aₙ ∩ N = {id} and AₙN = Sₙ. Thus Sₙ ≃ Aₙ × N by homework 8 problem 9. Hence N ⊂ Z(Sₙ) since N is commutative and commutes with all elements of Aₙ. This is a contradiction since Z(Sₙ) = {id}. So there is no normal subgroup of index 2 in Sₙ.

**Problem 8.5.10.** Since N ∩ Aₙ = Aₙ we have Aₙ ⊆ N. So by Lagrange’s theorem, the order of N is a multiple of |Aₙ| = |Sₙ|/2 dividing |Sₙ|, so it is either |Sₙ|/2 (in which case N = Aₙ) or |Sₙ| (in which case N = Sₙ).

**Problem 8.5.12.** Let f : Sₙ → Sₙ be a homomorphism of groups. Then for any 3-cycle σ, f(σ) has order either 3 or 1. If o(f(σ)) = 3, f(σ) is a product of disjoint 3-cycles, so f(σ) ∈ Aₙ. If o(f(σ)) = 1 then f(σ) = id ∈ Aₙ. Since the 3-cycles generate Aₙ we have f(Aₙ) ⊆ Aₙ.