Math 110AH Fall 18

HOMEWORK 8 - SOLUTIONS

Problem 1. (1234)(456)(145) = (152346).

Problems 2. This amounts to counting the number of r-cycles $(a_1 \dots a_r)$ in S_n. To construct an r-cycle, one has to choose an r-tuple $(a_1, \dots, a_r)$ of distinct elements in $\{1, \dots, n\}$, of which there are $n(n-1)\dots(n-r+1)$. Each r-cycle $(a_1 \dots a_r)$ corresponds to r possible r-tuples (one for each choice of the first element a_i appearing in the cycle). Hence there are $\frac{n(n-1)...(n-r+1)}{r}$ r-cycles in S_n.

Problem 3. Put $c = (12\dots r) \in S_n$, and let H be the set of elements of S_n commuting with c. Then H is the stabilizer of c under the action of S_n on itself by conjugation. From problem 2 we know that c has $\frac{n(n-1)...(n-r+1)}{r}$ conjugates in S_n. In other words, the orbit of c under the action of S_n by conjugation has size $\frac{n(n-1)...(n-r+1)}{r}$. So:

$$|H| = \frac{|S_n|}{|\text{orbit of } c|} = r(n-r)!$$

Let K be the set of elements of the form $c^i\tau$ with $i \in \mathbb{Z}$ and τ a permutation fixing $1, 2, \dots r$. Then clearly $K \subseteq H$, and K has $r(n-r)!$ elements. So $H = K$, and we are done.

Problem 4. Let $f \in S_n$ be an element commuting with $(12)(34)$. Then $(12)(34) = (f(1)f(2))(f(3)f(4))$. So we have the following possibilities:

- $\{f(1), f(2)\} = \{1, 2\}$ and $\{f(3), f(4)\} = \{3, 4\}$, then f has the form $f = (12)^i(34)^j\tau$ where τ fixes 1,2,3 and 4. There are $4(n-4)!$ elements of this form.
- $\{f(1), f(2)\} = \{3, 4\}$ and $\{f(3), f(4)\} = \{1, 2\}$, then f has the form $f = (13)(24)(12)^i(34)^j\tau$ where τ fixes 1,2,3 and 4. There are $4(n-4)!$ elements of this form.

We find that the number of conjugates of $(12)(34)$ in S_n is $\frac{n!}{8(n-4)!} = \frac{n(n-1)(n-2)(n-3)}{8}$.

Problem 5. Consider the elements $\tau = (12345)$ and $\tau' = (13245) = (23)\tau(23)$ of A_5. They are conjugate in S_5. However, they are not conjugate in S_5. Indeed, the subgroup H of elements of S_5 commuting with τ is $H = \langle\tau\rangle$, and in particular $H \subseteq A_5$. If τ and τ' were conjugate in A_5, we would have $\tau' = \sigma\tau\sigma^{-1}$ for some $\sigma \in A_5$, so $\tau' = (23)\tau(23) = \sigma\tau\sigma^{-1}$. This equality implies that $((23)\sigma)\tau((23)\sigma)^{-1} = \tau$, and so $(23)\sigma \in H \subseteq A_5$: this is impossible since $(23) \notin A_5$.

The general result is the following:

Theorem 1. *Let $\tau \in A_n$, and H be the subgroup of elements of S_n commuting with τ. If $H \nsubseteq A_n$, the conjugacy classes of τ in S_n and in A_n are the same. If not, the conjugacy class of τ in S_n is the union of two conjugacy classes in A_n: that of τ and that of $\epsilon\tau\epsilon^{-1}$, where ϵ is any odd permutation.*

Proof: exercise.

In A_5 we get have the following conjugacy classes:
- the class of e.
- the class of (123), containing all the 20 3-cycles of S_n since (45) is an odd permutation commuting with (123).
- the class of $(12)(34)$, containing all the 15 double transpositions of S_n since (12) is an odd permutation commuting with $(12)(34)$.
- the class of (12345), containing 12 elements (half of the 24 5-cycles of S_n).
- the class of (13245), containing the other 12 5-cycles of S_n.

We can check the class equation: $|A_5| = 60 = 1 + 20 + 15 + 12 + 12$.

Problem 6. The subgroups of A_4 are the following:
- the trivial group $\{e\}$.
- the 3 subgroups of order 2 generated by a double transposition: $\langle(12)(34)\rangle$, $\langle(13)(24)\rangle$ and $\langle(14)(23)\rangle$.
- the 4 subgroups of order 3 generated by 3-cycles: $\langle(123)\rangle$, $\langle(124)\rangle$, $\langle(134)\rangle$ and $\langle(234)\rangle$.
- the Klein group generated by all double transposition. It is isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.
- the whole group A_4.

Problem 7. We know that the order of an element of S_n is the lowest common multiple of the lengths of the disjoint cycles appearing in its decomposition. In S_5, the maximum number we can get is 6, it is the order of $(12)(345)$ for instance.

Problem 8. If $x \in S_p$ satisfies $x^p = e$, then either $x = e$ or x has order p. If x has order p, the cycles appearing in its decomposition have lengths whose lcm is p. Since p is prime, the only possibility in S_p is that x is a p-cycle.

Problem 9. Let f be an automorphism of S_3. Since (12) and (23) generate S_3, f is entirely determined by $f((12))$ and $f((23))$. Furthermore f preserves the order of elements, so $f((12))$ and $f((23))$ are two distinct transpositions of S_3. There are 3 transpositions in S_3, so there are 6 pairs of distinct transpositions. This proves that $|\mathrm{Aut}(S_3)| \leqslant 6$.

However, we know that $Z(S_3) = \{e\}$, so $6 = |S_3| = |\mathrm{Inn}(S_3)|$. Hence $\mathrm{Aut}(S_3) = \mathrm{Inn}(S_3)$.