Math 110AH Fall 18

HOMEWORK 7 - SOLUTIONS

Problem 1. The groups $(\mathbb{R}^*, \times)$ and $(\mathbb{C}^*, \times)$ are not isomorphic. Indeed, $\mathbb{R}^*$ does not have any element of order 3, while $\mathbb{C}^*$ has two: $e^{2i\pi/3}$ and $e^{4i\pi/3}$.

Problem 2. Let $m \in M$ and $n \in N$, and consider $x = mnm^{-1}n^{-1}$. Since M is normal, $nm^{-1}n^{-1} \in M$, so $x \in M$. Since N is normal, $mnm^{-1} \in N$ so $x \in N$. Hence $x \in M \cap N = \{e\}$. So $x = e$, and $mn = nm$.

Problem 3. The subgroup H is not normal if $n > 2$. Indeed, let f be the bijection that exchanges 2 and 3, and fixes every other element. Then $f \in H$. Let g be the bijection that exchanges 1 and 2, and fixes every other element. Then we have:

$$gfg^{-1}(1) = gf(2) = g(3) = 3.$$

So $gfg^{-1} \notin H$, and thus H is not normal.

Problem 4. Consider the map:

$$\alpha : \left\{ \begin{array}{ccc} G & \to & \mathrm{Inn}(G) \\ g & \mapsto & \alpha_g \end{array} \right.$$

where α_g is the automorphism of G defined by $\alpha_g(x) = gxg^{-1}$. By definition of $\mathrm{Inn}(G)$, α is surjective. Furthermore, α is a homomorphism since for all $g, h, x \in G$ we have:

$$\begin{array}{rcl} \alpha_g(\alpha_h(x)) & = & \alpha_g(hxh^{-1}) \\ & = & ghxh^{-1}g^{-1} \\ & = & (gh)x(gh)^{-1} \\ & = & \alpha_{gh}(x) \end{array}$$

The kernel of α is:

$$\begin{array}{rcl} \ker(\alpha) & = & \{g \in G \mid \alpha_g = \mathrm{id}_G\} \\ & = & \{g \in G \mid \forall x \in G,\, gxg^{-1} = x\} \\ & = & \{g \in G \mid \forall x \in G,\, gx = xg\} \\ & = & Z(G) \end{array}$$

Hence, α is a surjective homomorphism with kernel $Z(G)$. By the first isomorphism theorem, we get that $G/Z(G) \simeq \text{Inn}(G)$.

Examples of groups which have non inner automorphisms are $\mathbb{Z}/n\mathbb{Z}$ for $n > 2$. Indeed, $\mathbb{Z}/n\mathbb{Z}$ has $\Phi(n)$ automorphisms as seen in the previous homework, but only id is inner since $\mathbb{Z}/n\mathbb{Z}$ is abelian.

Problem 5. Consider the map:

$$f : \left\{ \begin{array}{ccc} \mathbb{C}^* & \to & \mathbb{C}^* \\ z & \mapsto & z^2 \end{array} \right.$$

The map f is a homomorphism since the multiplication in $\mathbb{C}^*$ is commutative. Furthermore, f is surjective. Indeed, given $z \in \mathbb{C}^*$, we can write z in polar form: $z = re^{i\theta}$ with $r > 0$ and $\theta \in \mathbb{R}$. Let $w = \sqrt{r}e^{i\theta/2}$, then $z = f(w)$.

Finally, the kernel of f is $\{\pm 1\} = H$. By the first isomorphism theorem, $\mathbb{C}^*/H \simeq \mathbb{C}^*$.

Problem 6. Consider the map:

$$f : \left\{ \begin{array}{ccc} \mathbb{R}^* & \to & \mathbb{R}_{>0} \\ x & \mapsto & x^2 \end{array} \right.$$

The map f is a homomorphism since the multiplication in $\mathbb{R}^*$ is commutative. Furthermore, f is surjective. Indeed, given $y \in \mathbb{R}_{>0}$, we have $y = f(\sqrt{y})$. Finally, the kernel of f is $\{\pm 1\} = H$. By the first isomorphism theorem, $\mathbb{R}^*/H \simeq \mathbb{R}_{>0}$.

12.11 - Problem 7. Consider the map:

$$\theta : \left\{ \begin{array}{ccc} N_G(H) & \to & \text{Aut}(H) \\ x & \mapsto & \theta_x \end{array} \right.$$

where θ_x is the automorphism of H defined by $\theta_x(h) = xhx^{-1}$ (this is well defined because H is normal in $N_G(H)$). By the same argument as problem 4, θ is a homomorphism, and its kernel is $Z_G(H)$ by definition. In particular, $Z_G(H)$ is a normal subgroup of $N_G(H)$ and θ induces a map $N_G(H)/Z_G(H) \to \text{Aut}(H)$.

13.6 - Problem 4. Consider the homomorphism:

$$u : \left\{ \begin{array}{ccc} G & \to & (G/H) \times (G/N) \\ g & \mapsto & (Hg, Ng) \end{array} \right.$$

The kernel of u is $H \cap N$. Let us prove that u is surjective. Consider $x = (Ha, Nb) \in (G/H) \times (G/N)$. Since $G = HN$, we have $ab^{-1} = hn$ for some $h \in H$ and $n \in N$. Then

$Ha = Hnb$, and $Nb = Nnb$, so $x = u(nb)$. Hence u is surjective. By the first isomorphism theorem, there is an isomorphism:

$$G/(H \cap N) \simeq (G/H) \times (G/N)$$

Problem 8. We proceed by induction on $|G|$.

- If $|G| = p$, G is cyclic and has an element of order p.
- Assume the result is true for all groups of order $< N$ and let G be a group of order $N+1$. Let $a \in G$, $a \neq e$, and $n = o(a) > 1$. If $p|n$ then $a^{n/p}$ has order p and we are done. Otherwise, consider the group $G/\langle a \rangle$, it has order $\frac{N+1}{n} < N$, and since $p|(N+1)$ and p is coprime with n, $p|\frac{N+1}{n}$. By induction we can find an element $\langle a \rangle g$ of order p, for some $g \in G$. In particular, $p|o(g)$. So $\langle g \rangle$ contains an element of order p, and we are done.

Problem 9. Let F be a finite subgroup of order n of $\mathbb{Q}/\mathbb{Z}$. Then by the previous homework, $F = \langle \frac{1}{n} + \mathbb{Z} \rangle$. Consider the homomorphism:

$$f : \left\{ \begin{array}{ccc} \mathbb{Q}/\mathbb{Z} & \to & \mathbb{Q}/\mathbb{Z} \\ x & \mapsto & nx \end{array} \right.$$

Given $y = \frac{p}{q} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$, we have $y = f\left(\frac{p}{nq} + \mathbb{Z}\right)$. Hence f is surjective.

Let us determine $\ker(f)$. It is clear that $f\left(\frac{1}{n} + \mathbb{Z}\right) = 0$, so $F \subseteq \ker(f)$. Conversely, let $\frac{p}{q} + \mathbb{Z} \in \ker(f)$, with p, q two coprime integers. Then $\frac{np}{q} \in \mathbb{Z}$. So $q|np$, and since q and p are coprime, $q|n$. Writing $n = pu$ for some $u \in \mathbb{Z}$, we see that $\frac{p}{q} = pu\frac{1}{n}$, so $\frac{p}{q} + \mathbb{Z} \in F$. So $\ker(f) = F$.

By the first isomorphism theorem, $(\mathbb{Q}/\mathbb{Z})/F \simeq \mathbb{Q}/\mathbb{Z}$.

Problem 10. Let G be a group of order p^2. By Cauchy's theorem, G has an element g of order p. Let $H = \langle g \rangle$, and consider the Cayley homomorphism $G \to \mathrm{Bij}(G/H)$. Recall that its kernel N is the maximal normal subgroup contained in H. Since $|H| = p$, $N = H$ or $N = \{e\}$.

However, $|G| = p^2$, and $|\mathrm{Bij}(G/H)| = p!$. Since p^2 does not divide $p!$, the Cayley homomorphism is not injective and $N \neq \{e\}$. So $N = H$, and H is a normal subgroup of order p.