Math 110B Winter 18

HOMEWORK 7 - SOLUTIONS

Problem 8.4.16. Note that this function is just the square of the module: for all $z \in \mathbb{C}^*$, we have $f(z) = |z|^2$. Hence for all $z, w \in \mathbb{C}^*$, we have $f(zw) = |zw|^2 = (|z||w|)^2 = |z|^2|w|^2 = f(z)f(w)$. So $f$ is a homomorphism of groups. Furthermore, if $r \in \mathbb{R}^{**}$, we have $r = f(\sqrt{r})$, so $f$ is surjective.

Problem 8.4.18. Let us start by finding all the normal subgroups of $D_4$. Denote by $r$ the rotation of angle $\pi/2$. For any reflection $s$ of $D_4$, we know that $r$ and $s$ generate $D_4$, and we have $s^2 = r^4 = e$ and $srs = r^{-1}$. Note that since $r, s$ generate $D_4$, to check that a subgroup $H$ is normal it suffices to check that $rHr^{-1} \subseteq H$ and $sHs^{-1} \subseteq H$. There are two types of subgroups of $D_4$:

- those contained in $\langle r \rangle \simeq \mathbb{Z}_4$. By the classification of subgroups of cyclic groups, we know that there exactly 3 of those: $\{e\}$, $\langle r^2 \rangle$, and $\langle r \rangle$. Using the formula $sr^k s = r^{-k}$, it follows that they are all normal. The corresponding quotients are $D_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_2$.
- those containing a reflection $s$. Let $H$ be such a subgroup. If $H = \langle s \rangle$, $H$ is not normal because $rsr^{-1} = r^2s \not\subseteq H$. If $H \neq \langle s \rangle$, $H$ contains an element of the form $r^k s$ for $k = 1, 2, 3$. Hence $H$ contains $r^k$, as we see by multiplying on the right by $s$. If $k = 1, 3$, $H = D_4$, so $H$ is normal and the corresponding quotient is $\{e\}$. If $k = 2$ and $H$ does not contain $r$, we have $H = \{e, r^2, s, r^2s\}$. Then $H$ is normal since we have:

$$sr^2s = r^2 \in H, \quad rsr^{-1} = r^2s \in H$$

In that case, the quotient is $\mathbb{Z}_2$.

In conclusion, the homomorphic images of $D_4$ are $\{e\}, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2$ and $D_4$.

Problem 8.4.21. Let $K = \ker(f)$, we know that $K$ is a normal subgroup $G$. Since $G$ is simple, we have either $K = G$ or $K = \{e\}$. If $K = G$, we have $\text{Im}(f) = \{e\}$ so $H = \{e\}$ since $f$ is surjective. If $K = \{e\}$, $f$ is injective, since it is also surjective it is an isomorphism.

Problem 8.4.22. (a) The identity element $e$ has order 1, so $e \in K$ and $K$ is not empty. Let $x, y \in K$. Then we have $(xy^{-1})^2 = x^2y^{-2} = e$, the first equality holding since $G$ is abelian. Then we have $o(xy^{-1}) \leqslant 2$, so $xy^{-1} \in K$. Hence $K$ is a subgroup of $G$. 

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(b) Since $G$ is not empty, we see that $H$ is not empty from its definition. Let $a, b \in H$, we have $a = x^2$ and $b = y^2$ for some $x, y \in G$. Then $ab^{-1} = (xy-1)^2$ since $G$ is abelian, so $ab^{-1} \in H$. Thus $H$ is a subgroup of $G$.

(c) Consider the map:

$$
\left\{ \begin{array}{ccc}
G & \rightarrow & H \\
x & \mapsto & x^2
\end{array} \right.
$$

Since $G$ is abelian, for all $x, y \in G$ we have $(xy)^2 = x^2y^2$, so $f$ is a homomorphism. Then, from their respective definitions we see that $K = \ker(f)$ and $H = \text{Im}(f)$. It follows from the first isomorphism theorem that $G/K \simeq H$.

**Problem 8.4.24.** Since $k|n$, the (proof of the) third isomorphism theorem guarantees that the map $\mathbb{Z}_n \rightarrow \mathbb{Z}_k$, $[x]_n \mapsto [x]_k$ is a well defined homomorphism of additive groups. It is actually easy to see that it is a homomorphism of rings from the definition of the multiplication in both rings. It follows that it restricts to a homomorphism of groups between the groups of units.

**Problem 8.4.29.** We use the third isomorphism theorem : since $k|n$, we have $n\mathbb{Z} \subseteq k\mathbb{Z}$, and so $(\mathbb{Z}/n\mathbb{Z})/(k\mathbb{Z}/n\mathbb{Z}) \simeq \mathbb{Z}/k\mathbb{Z}$, or written with our usual notations $\mathbb{Z}_n/\langle k \rangle \simeq \mathbb{Z}_k$.

**Problem 8.4.34.** (a) For simplification, for $a, b, c \in \mathbb{Q}$, let $[a, b, c]$ denote the matrix defined the problem. It is clear that $G$ is not empty, as it contains $I_3 = [0, 0, 0]$ for instance. We have:

$$
[a, b, c] [\alpha, \beta, \gamma] = [a + \alpha, \beta + b + a\gamma, c + \gamma] \quad (*)
$$

from which stability under multiplication follows. Using this formula, we can see that $[a, b, c]^{-1} = [-a, ac - b, -c]$. So $G$ is stable by inverse. Hence $G$ is a subgroup of $\text{GL}_3(\mathbb{Q})$, and in particular a group under matrix multiplication.

(b) Using the formula $(*)$ from above, we see that $[a, b, c] \in Z(G)$ if and only if for all $a, \beta, \gamma \in \mathbb{Q}$ we have:

$$
\left\{ \begin{array}{ccc}
a + a &=& a + a \\
\beta + a\gamma + b &=& b + ac + \beta \\
b + \beta &=& \beta + b
\end{array} \right.
$$

which simplify into the unique condition $a\gamma = ac$. This holds for all $a, \gamma$ if and only if $a = c = 0$. It follows that $Z(G) = \{[0, b, 0], b \in \mathbb{Q}\}$.

From this description we see that we have a bijection $\mathbb{Q} \rightarrow Z(G), b \mapsto [0, b, 0]$. Furthermore, $(*)$ shows that this is a homomorphism of groups. Hence $Z(G)$ is isomorphic to $\mathbb{Q}$. 

(c) Consider the map:
\[
\begin{array}{ccc}
G & \rightarrow & Q \times Q \\
[a,b,c] & \mapsto & (a,c)
\end{array}
\]
By (\ast), this is a homomorphism of groups. This is clearly surjective. By (b), the kernel of this homomorphism is C. Hence by the first isomorphism theorem, \( G/C \cong Q \times Q \).

**Problem 8.4.41.** (a) Let \( a \in G \). Then for all \( b \in G \), we have:
\[
(f_a \circ f_{a^{-1}})(Kb) = f_a(Kba^{-1}) = Kba^{-1}a = Kb
\]
Hence \( f_a \circ f_{a^{-1}} = \text{id} \). Applying this to \( a^{-1} \) gives \( f_{a^{-1}} \circ f_a = \text{id} \) since \( (a^{-1})^{-1} = a \). Thus \( f_a \) is a permutation of \( T \).

(b) Let \( g, h \in G \), and \( b \in G \). Then we have:
\[
(f_g^{-1} \circ f_{h^{-1}})(Kb) = f_{g^{-1}}(Kbh^{-1}) = Kbh^{-1}g^{-1} = Kb(gh)^{-1} = f_{(gh)^{-1}}
\]
Thus \( \phi(gh) = \phi(g) \circ \phi(h) \), and \( \phi \) is a homomorphism of groups.

If \( g \in \ker(\phi) \), then in particular we have \( f_{g^{-1}}(K) = K \). This gives \( K_{g^{-1}} = K \), thus \( g \in K \). So \( \ker(\phi) \subseteq K \).

(c) Assume \( K \) is normal, and let \( k \in K \). Then for all \( b \in G \) we have \( bk^{-1}b^{-1} \in K \), so \( K = Kbk^{-1}b^{-1} \). It follows that \( Kbk^{-1} = Kbk^{-1}b^{-1}b = Kb \), so \( \phi(k)(Kb) = Kb \). Hence \( \phi(k) = \text{id} \). So we have \( K = \ker(\phi) \).

(d) Taking \( K = \{e\} \), the previous result tells us that we have an injection of \( G \) into \( A(G) \), which is exactly the statement of Cayley’s theorem.

**Problem 8.4.42.** (a) Consider the (unique) subgroup \( N \) of \( S_3 \) generated by an element of order 3. Since it is cyclic, it is abelian. Since it has index 2, it is normal, with quotient \( S_3/N \cong \mathbb{Z}_2 \) abelian. Hence \( S_3 \) is metabelian.

(b) Let \( G \) be a metabelian group, and \( H \) a normal subgroup of \( G \). We want to prove that \( G/H \) is metabelian. Let \( N \) be a normal subgroup such that \( N \) and \( G/N \) are abelian. Let \( K \) be the image of \( N \) in \( G/H \). This is an abelian subgroup of \( G/H \) since it is the image of an abelian subgroup by a homomorphism. It is also normal because it is the image of a normal subgroup by a surjective homomorphism (as shown in previous homework).

By the third isomorphism theorem, \( (G/H)/K \) is a quotient of \( G/N \), so in particular it is abelian. Hence \( G/H \) is metabelian.

(c) Let \( G \) be a metabelian group, and \( H \) a normal subgroup of \( G \). We want to prove that \( H \) is metabelian. Let \( N \) be a normal subgroup such that \( N \) and \( G/N \) are abelian. Let \( K = H \cap N \), then \( K \) is an abelian normal subgroup of \( H \). Furthermore, by the second isomorphism theorem, \( H/K \) is a subgroup of \( G/N \), so in particular it is abelian. Hence \( H \)
is metabelian.

**Problem 7.5.1.** (a) (173). (b) (1245789). (c) (1476283). (d) (35798).


**Problem 7.5.9.** (a) 12. (b) 60. (c) 10!/2.

**Problem 7.5.11.** The elements of $A_4$ are:
- the identity element, which has order 1.
- the 8 3-cycles, which have order 3.
- the 3 products of two disjoint 2-cycles, which have order 2.

**Problem 7.5.12.** (12)(34) = (123)(234).

**Problem 7.5.13.** The decomposition of $\alpha$ as a product of disjoint cycles is $\alpha = (12)(34)(56789(10))$, so by theorem 7.25 it has order $[2, 2, 6] = 6$.

**Problem 7.5.14.** The decomposition of $\beta$ as a product of disjoint cycles is $\beta = (1236784)(59(10))$, so by theorem 7.25 it has order $[7, 3] = 21$. 