Problem 1. The groups \((\mathbb{R}^*, \times)\) and \((\mathbb{C}^*, \times)\) are not isomorphic. Indeed, \(\mathbb{R}^*\) does not have any element of order 3, while \(\mathbb{C}^*\) has two: \(e^{2i\pi/3}\) and \(e^{4i\pi/3}\).

Problem 2. Let \(m \in M\) and \(n \in N\), and consider \(x = mnm^{-1}n^{-1}\). Since \(M\) is normal, \(mnm^{-1}n^{-1} \in M\), so \(x \in M\). Since \(N\) is normal, \(mnm^{-1} \in N\) so \(x \in N\). Hence \(x \in M \cap N = \{e\}\). So \(x = e\), and \(mn = nm\).

Problem 3. The subgroup \(H\) is not normal if \(n > 2\). Indeed, let \(f\) be the bijection that exchanges 2 and 3, and fixes every other element. Then \(f \in H\). Let \(g\) be the bijection that exchanges 1 and 2, and fixes every other element. Then we have:

\[
gfg^{-1}(1) = gf(2) = g(3) = 3.
\]

So \(gfg^{-1} \notin H\), and thus \(H\) is not normal.

Problem 4. Consider the map:

\[
\alpha : \begin{cases}
G & \to \text{Inn}(G) \\
g & \mapsto \alpha_g
\end{cases}
\]

where \(\alpha_g\) is the automorphism of \(G\) defined by \(\alpha_g(x) = gxg^{-1}\). By definition of \(\text{Inn}(G)\), \(\alpha\) is surjective. Furthermore, \(\alpha\) is a homomorphism since for all \(g, h, x \in G\) we have:

\[
\alpha_g(\alpha_h(x)) = \alpha_g(hxh^{-1})
\]

\[
= ghxh^{-1}g^{-1}
\]

\[
= (gh)x(gh)^{-1}
\]

\[
= \alpha_{gh}(x)
\]

The kernel of \(\alpha\) is:

\[
\ker(\alpha) = \{g \in G | \alpha_g = \text{id}_G\}
\]

\[
= \{g \in G | \forall x \in G, gxg^{-1} = x\}
\]

\[
= \{g \in G | \forall x \in G, gx = xg\}
\]

\[
= Z(G)
\]
Hence, $\alpha$ is a surjective homomorphism with kernel $Z(G)$. By the first isomorphism theorem, we get that $G/Z(G) \simeq \text{Inn}(G)$.

Examples of groups which have non inner automorphisms are $\mathbb{Z}/n\mathbb{Z}$ for $n > 2$. Indeed, $\mathbb{Z}/n\mathbb{Z}$ has $\Phi(n)$ automorphisms as seen in the previous homework, but only id is inner since $\mathbb{Z}/n\mathbb{Z}$ is abelian.

**Problem 5.** Consider the map:

$$f : \{ \mathbb{C}^* \to \mathbb{C}^* \}$$

The map $f$ is a homomorphism since the multiplication in $\mathbb{C}^*$ is commutative. Furthermore, $f$ is surjective. Indeed, given $z \in \mathbb{C}^*$, we can write $z$ in polar form: $z = re^{i\theta}$ with $r > 0$ and $\theta \in \mathbb{R}$. Let $w = \sqrt{r}e^{i\theta/2}$, then $z = f(w)$.

Finally, the kernel of $f$ is $\{ \pm 1 \} = H$. By the first isomorphism theorem, $\mathbb{C}^*/H \simeq \mathbb{C}^*$.

**Problem 6.** Consider the map:

$$f : \{ \mathbb{R}^* \to \mathbb{R}_{>0} \}$$

The map $f$ is a homomorphism since the multiplication in $\mathbb{R}^*$ is commutative. Furthermore, $f$ is surjective. Indeed, given $y \in \mathbb{R}_{>0}$, we have $y = f(\sqrt{y})$. Finally, the kernel of $f$ is $\{ \pm 1 \} = H$. By the first isomorphism theorem, $\mathbb{R}^*/H \simeq \mathbb{R}_{>0}$.

**12.11 - Problem 7.** Consider the map:

$$\theta : \{ N_G(H) \to \text{Aut}(H) \}$$

where $\theta_x$ is the automorphism of $H$ defined by $\theta_x(h) = xhx^{-1}$ (this is well defined because $H$ is normal in $N_G(H)$). By the same argument as problem 4, $\theta$ is a homomorphism, and its kernel is $Z_G(H)$ by definition. In particular, $Z_G(H)$ is a normal subgroup of $N_G(H)$ and $\theta$ induces a map $N_G(H)/Z_G(H) \to \text{Aut}(H)$.

**13.6 - Problem 4.** Consider the homomorphism:

$$u : \{ G \to (G/H) \times (G/N) \}$$

The kernel of $u$ is $H \cap N$. Let us prove that $u$ is surjective. Consider $x = (Ha, Nb) \in (G/H) \times (G/N)$. Since $G = HN$, we have $ab^{-1} = hn$ for some $h \in H$ and $n \in N$. Then
\[ Ha = Hnb, \text{ and } Nb = Nnb, \text{ so } x = u(nb). \text{ Hence } u \text{ is surjective. By the first isomorphism theorem, there is an isomorphism:} \]
\[ G / (H \cap N) \cong (G / H) \times (G / N) \]

**Problem 8.** We proceed by induction on \(|G|\).

- If \(|G| = p\), \(G\) is cyclic and has an element of order \(p\).
- Assume the result is true for all groups of order \(< N\) and let \(G\) be a group of order \(N + 1\). Let \(a \in G, a \neq e, \text{ and } n = o(a) > 1\). If \(p \mid n\) then \(a^n / p\) has order \(p\) and we are done. Otherwise, consider the group \(G / \langle a \rangle\), it has order \(\frac{N + 1}{n}\), and since \(p \mid (N + 1)\) and \(p\) is coprime with \(n, p \mid \frac{N + 1}{n}\). By induction we can find an element \(\langle a \rangle g\) of order \(p\), for some \(g \in G\). In particular, \(p \mid o(g)\). So \(\langle g \rangle\) contains an element of order \(p\), and we are done.

**Problem 9.** Let \(F\) be a finite subgroup of order \(n\) of \(\mathbb{Q} / \mathbb{Z}\). Then by the previous homework, \(F = \langle \frac{1}{n} + \mathbb{Z} \rangle\). Consider the homomorphism:
\[ f: \begin{cases} 
\mathbb{Q} / \mathbb{Z} & \rightarrow \mathbb{Q} / \mathbb{Z} \\
x & \mapsto nx 
\end{cases} \]
Given \(y = \frac{p}{q} + \mathbb{Z} \in \mathbb{Q} / \mathbb{Z}\), we have \(y = f \left( \frac{p}{aq} + \mathbb{Z} \right)\). Hence \(f\) is surjective.

Let us determine \(\ker(f)\). It is clear that \(f \left( \frac{1}{n} + \mathbb{Z} \right) = 0\), so \(F \subseteq \ker(f)\). Conversely, let \(\frac{p}{q} + \mathbb{Z} \in \ker(f)\), with \(p, q\) two coprime integers. Then \(\frac{np}{q} \in \mathbb{Z}\). So \(q \mid np\), and since \(q\) and \(p\) are coprime, \(q \mid n\). Writing \(n = pu\) for some \(u \in \mathbb{Z}\), we see that \(\frac{p}{q} = pu\frac{1}{n}\), so \(\frac{p}{q} + \mathbb{Z} \in F\). So \(\ker(f) = F\).

By the first isomorphism theorem, \((\mathbb{Q} / \mathbb{Z}) / F \cong \mathbb{Q} / \mathbb{Z}\).

**Problem 10.** Let \(G\) be a group of order \(p^2\). By Cauchy’s theorem, \(G\) has an element \(g\) of order \(p\). Let \(H = \langle g \rangle\), and consider the Cayley homomorphism \(G \to \Bij(G / H)\). Recall that its kernel \(N\) is the maximal normal subgroup contained in \(H\). Since \(|H| = p, N = H\) or \(N = \{e\}\).

However, \(|G| = p^2\), and \(|\Bij(G / H)| = p!\). Since \(p^2\) does not divide \(p!\), the Cayley homomorphism is not injective and \(N \neq \{e\}\). So \(N = H, \text{ and } H\) is a normal subgroup of order \(p\).