Math 110B Winter 18

HOMEWORK 6 - SOLUTIONS

Problem 8.3.8. Consider the homomorphism of groups:

\[
\begin{array}{c}
\mathbb{Z}_4 \times \mathbb{Z}_4 \rightarrow \mathbb{Z}_4 \\
(a, b) \mapsto b - 2a
\end{array}
\]

It is clearly surjective, and it is easy to check that its kernel is \( N \). From the first isomorphism theorem, we deduce that \( \mathbb{Z}_4 \times \mathbb{Z}_4 / N \simeq \mathbb{Z}_4 \).

Problem 8.3.9. The element \((1, 1)\) has order 6 in \( \mathbb{Z}_6 \times \mathbb{Z}_2 \). Hence, \( N \) has index 2. Since up to isomorphism the only group of order 2 is \( \mathbb{Z}_2 \), we deduce that \( \mathbb{Z}_6 \times \mathbb{Z}_2 / N \simeq \mathbb{Z}_2 \).

Problem 8.3.10. (a) In \( \mathbb{Z}_2 \times \mathbb{Z}_4 \), the elements \((0, 2)\) and \((1, 2)\) have the same order (which is 2). Hence, the cyclic subgroups they generate are isomorphic.

(b) Using the definition of the operation in quotient groups, we find that the operation table of \( G / M \) is:

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<th>( M + (0, 0) )</th>
<th>( M + (1, 0) )</th>
<th>( M + (0, 1) )</th>
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(c) From the table we have just found, we deduce that \( G / M \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \). However, from the table given in example 4, we see that \( G / N \simeq \mathbb{Z}_4 \). Hence \( G / M \) and \( G / N \) are not isomorphic, even though \( M \) and \( N \) are.

Problem 8.3.19. Let \( g \in G \). Since the elements of \( G / N \) are all square by assumption, \( Ng \) is square, so we have \( Ng = (Nh)^2 \) for some \( h \in G \). By definition of the operation in \( G / N \), this can be written \( Ng = Nh^2 \). This means that \( gh^{-2} \in N \). Since all the elements of \( N \) are assumed to be square, we can find \( n \in N \) such that \( gh^{-2} = n^2 \), which gives \( g = n^2h^2 \). Since \( G \) is abelian, we have \( n^2h^2 = (nh)^2 \). Hence \( g = (nh)^2 \) is square. Thus every element of \( G \) is square.
Problem 8.3.25. (a) Let us do something a little more general and prove that if \( p \in \mathbb{Z} \) and \( q \in \mathbb{N} \setminus \{0\} \) are relatively prime, then the order of \( r = \mathbb{Z} + \frac{p}{q} \) in \( \mathbb{Q} / \mathbb{Z} \) is \( q \). First, note that \( qr = \mathbb{Z} + p = 0 \). Let \( n \geq 1 \) be such that \( nr = 0 \), let us prove that \( n \geq q \). The equality \( nr = 0 \) means that \( npq \in \mathbb{Z} \), which means that \( q | np \). Since \( q \) and \( p \) are relatively prime, we get that \( q | n \). Hence the order of \( r \) is exactly \( q \).

Now we can use this to deduce that the order of \( \frac{8}{9} \) is 9, that of \( \frac{14}{5} \) is 5, and that of \( \frac{48}{28} = \frac{12}{7} \) is 7.

(b) Every element of \( \mathbb{Q} \) can be written \( \frac{p}{q} \) for some \( p \in \mathbb{Z} \) and \( q \in \mathbb{N} \setminus \{0\} \) relatively prime. Thus, every element of \( \mathbb{Q} / \mathbb{Z} \) can be written in the form \( \mathbb{Z} + \frac{p}{q} \) for some \( p \in \mathbb{Z} \) and \( q \in \mathbb{N} \setminus \{0\} \) relatively prime. Using what we have shown in part (a), it follows that every element of \( \mathbb{Q} / \mathbb{Z} \) has finite order.

(c) Let \( n \) be an integer \( \geq 1 \). By what we have proved in part (a), \( \mathbb{Z} + \frac{1}{n} \) has order \( n \) in \( \mathbb{Q} / \mathbb{Z} \). Hence there are elements in \( \mathbb{Q} / \mathbb{Z} \) of any possible finite order \( n \geq 1 \).

Problem 8.3.27. (a) Consider the following map:

\[
\begin{align*}
\mathbf{u} : & \quad G \rightarrow G^* \\
& \quad a \mapsto (a,e)
\end{align*}
\]

It is clearly a bijection. Furthermore, for all \( a, b \in G \) we have \( \mathbf{u}(ab) = (ab,e) = (a,e)(b,e) = \mathbf{u}(a)\mathbf{u}(b) \). Hence \( F \) is an isomorphism. So \( G^* \) is a subgroup of \( G \times H \) isomorphic to \( G \).

(b) Let \( (a,g) \in G^* \), and \( (g,h) \in G \times H \). Then we have:

\[
(g,h)(a,e)(g,h)^{-1} = (gag^{-1},heh^{-1}) = (gag^{-1},e) \in G^*
\]

Thus \( G^* \) is a normal subgroup of \( G \times H \).

(c) Consider the map:

\[
\begin{align*}
\mathbf{v} : & \quad H \rightarrow (G \times H) / G^* \\
& \quad h \mapsto G^*(e,h)
\end{align*}
\]

For all \( h, h' \in H \), we have:

\[
\mathbf{v}(hh') = G^*(e, hh') = G^*((e,h)(e,h')) = (G^*(e,h))(G^*(e,h')) = \mathbf{v}(h)\mathbf{v}(h')
\]
so \( v \) is a homomorphism of groups. Furthermore if \( v(h) = v(h') \), we have \( G^*(e,h) = G^*(e,h') \) which means that \( (e,h'h^{-1}) \in G^* \). Thus \( h'h^{-1} = e \) and \( h' = h \). So \( v \) is also injective. Finally, let \( x = G^*(a,b) \) be an element of \( (G \times H) / G^* \). Since \( (a,b)(e,b)^{-1} = (a,e) \in G^* \), we have \( x = G^*(e,b) = v(b) \). So \( v \) is surjective, hence an isomorphism. So \( (G \times H) / G^* \) is isomorphic to \( H \).

**Problem 8.3.29.** Let \( g \in G \). By assumption, every element of \( G/N \) has finite order, so there exists \( k \geq 1 \) such that \( (Ng)^k = e \). We can write this \( Ng^k = e \) by definition of the product in \( G/N \). Then this equality means that \( g^k \in N \). Since every element of \( N \) is assumed to have finite order, there exists \( l \geq 1 \) such that \( (g^k)^l = e \), i.e. \( g^{kl} = e \); \( g \) has finite order. Hence every element of \( G \) has finite order.

**Problem 8.3.33.** (a) Since \( G' \) is generated by \( S \), it suffices to check that if \( s \in S \) and \( g \in G \), \( gsg^{-1} \in G' \). We have \( s = aba^{-1}b^{-1} \) for some \( a, b \in G \). Then:

\[
gsg^{-1} = gaba^{-1}b^{-1}g^{-1} \\
= (gag^{-1})(gbg^{-1})(ga^{-1}g^{-1})(gb^{-1}g^{-1}) \\
= (gag^{-1})(gbg^{-1})(gag^{-1})^{-1}(gb^{-1}g^{-1})^{-1} \\
\in S
\]

Hence \( G' \) is a normal subgroup of \( G \).

(b) Let \( x, y \in G/G' \). Then we have \( x = G'a \) and \( y = G'b \) for some \( a, b \in G \). Thus we have \( xyx^{-1}y^{-1} = G'(aba^{-1}b^{-1}) \) by definition of the operation in \( G/G' \). Since \( aba^{-1}b^{-1} \in G' \), we get \( xyx^{-1}y^{-1} = e \) in \( G/G' \). Thus \( G/G' \) is abelian.

**Problem 8.3.34.** (a) First, we have \( (0,0) \in N \) so \( N \) is not empty. Then, if \( (x,y), (z,t) \in N \), we have \( y = -x \) and \( t = -z \). So \( y - t = -(x - z) \), which means that \( (x,y) = (z,t) \in N \). Hence \( N \) is a subgroup of \( \mathbb{R} \times \mathbb{R} \).

(b) Let us prove that the map:

\[
f : \begin{cases} \\
\mathbb{R} & \rightarrow (\mathbb{R} \times \mathbb{R})/N \\
x & \mapsto N + (x,0) \\
\end{cases}
\]

is an isomorphism. It is a homomorphism by the same arguments used in the previous problems. It is injective because if \( N + (x,0) = N + (y,0) \), we have \( (x - y, 0) \in N \), hence \( x - y = 0 \). Finally, if we have an element \( N + (a,b) \) in \( (\mathbb{R} \times \mathbb{R})/N \), we have \( (a,b) - (a + b, 0) \in N \), so \( N + (a,b) = N + (a + b, 0) = f(a + b) \). Hence \( f \) is also surjective. In conclusion, we have \( (\mathbb{R} \times \mathbb{R})/N \cong \mathbb{R} \).
Problem 8.4.2. $g$ is a homomorphism. Indeed, if $x, y \in \mathbb{R}$, we have:

- if $x, y$ are of the same sign, then $xy$ is positive so $g(xy) = 0$. On the other hand $g(x) = g(y)$, so $g(x) + g(y) = 2g(x) = 0$. Hence $g(xy) = g(x) + g(y)$.
- if $x$ and $y$ are of opposite signs, then $xy$ is negative so $g(xy) = 1$. On the other hand $g(x)$ and $g(y)$ are the two distinct elements of $\mathbb{Z}_2$, so $g(x) + g(y) = 0 + 1 = 1$. Hence $g(xy) = g(x) + g(y)$.

In all cases, we have $g(xy) = g(x) + g(y)$. The kernel of $g$ is clear by definition: it is the subgroup $\mathbb{R}_{>0}$ of positive real numbers.

Problem 8.4.4. For all $x, y \in \mathbb{Q}^*$ we have $|xy| = |x||y|$. Hence $f$ is a homomorphism. Its kernel is the set of $x \in \mathbb{Q}^*$ such that $|x| = 1$, so $\ker(f) = \{\pm 1\}$.

Problem 8.4.6. This is not a homomorphism. Indeed, we have $h(1 + 1) = 2^4 = 16$, but $h(1) + h(1) = 1^4 + 1^4 = 2$.

Problem 8.4.8. The fact that $f$ is a homomorphism follows from the distributivity axiom in the ring $\mathbb{Z}_{12}$. Let $x \in \mathbb{Z}$ such that $f([x]_{12}) = 0$. Then we must have $12|3x$, so $4|x$. It follows that $\ker(f) \subseteq \langle 4 \rangle$. Conversely, $f(4) = 0$, so $\langle 4 \rangle \subseteq \ker(f)$. In conclusion $\ker(f) = \langle 4 \rangle$.

Problem 8.4.10. Let us prove that $\phi$ is a homomorphism. For $\sigma, \tau \in S_n$ and $k \in \{1, \ldots, n\}$ we have $(\phi(\sigma) \circ \phi(\tau))(k) = \phi(\sigma)(\tau(k)) = \sigma(\tau(k))$, since $\tau(k) \in \{1, \ldots, n\}$. Also, $(\phi(\sigma) \circ \phi(\tau))(n + 1) = \phi(\sigma)(n + 1) = n + 1$. Hence, $\phi(\sigma) \circ \phi(\tau) = \phi(\sigma \circ \tau)$.

Let us show that $\ker(\phi) = \{\text{id}\}$. Let $\sigma \in \ker(\phi)$. Then for all $k \in \{1, \ldots, n\}$ we have $\sigma(k) = \phi(\sigma)(k) = k$ since $\phi(\sigma) = \text{id}$. Hence $\sigma = \text{id}$. 