**Problem 1.** Denote by $S$ the set of subgroups of $G$, which is by assumption finite. Consider the map:
$$f : \{G \rightarrow S\} : g \mapsto \langle g \rangle$$
If $H \in f(G)$, $H$ is a cyclic group. In particular $H$ has finitely many generators, so $f^{-1}(H)$ is finite. We have:
$$G = \bigsqcup_{H \in S} f^{-1}(H)$$
So $G$ is a finite union of finite sets, and thus is finite.

**Problem 2.** Assume $K \cup H$ is a subgroup, and $K \nsubseteq H$. We need to prove that $H \subseteq K$. Fix an element $k \in K \setminus H$. Given $h \in H$, we have $kh \in K \cup H$ since $K \cup H$ is a subgroup. However, $kh \notin H$ (otherwise, $k = khh^{-1} \in H$). Thus $kh \in K$. It follows that $h = k^{-1}kh \in K$. This holds for all $h \in H$, so $H \subseteq K$, which is what we wanted.

Conversely, if $K \subseteq H$ or $H \subseteq K$, we have $K \cup H = H$ or $K \cup H = K$, thus $K \cup H$ is a subgroup.

**Problem 3.** We start by decomposing $H$ as a union of $H \cap K$ cosets:
$$H = \bigsqcup_{h \in X} h \bigrangle H \cap K\bigrrangle$$
where $X$ denotes a set of representatives of classes in $H$ modulo $H \cap K$. Given $h, h' \in H$, note that:
$$h^{-1}h' \in H \cap K \iff h^{-1}h' \in K$$
So $h \bigrangle (H \cap K) = (hK) \cap H$. Since there are finitely many $K$ cosets in $G$, we deduce that there are finitely many $H \cap K$ cosets in $H$: $X$ is finite.

We can decompose $G$ as a union of $H$ cosets:
$$G = \bigsqcup_{g \in Y} gH$$
where $Y$ denotes a set of representatives of classes in $G$ modulo $H$ (in particular $Y$ is finite). For all $g \in Y$, we have:

$$gH = \bigsqcup_{h \in X} gh(H \cap K)$$

So we deduce:

$$G = \bigsqcup_{(g,h) \in X \times Y} gh(H \cap K)$$

Since $X$ and $Y$ are finite, $H \cap K$ has finite index in $G$.

**Problem 4.** Let $x \in G$, Since $e \in A, B$ and $x = exe$, we have $x \sim x$. So the relation is reflexive.

Let $x, y \in G$ such that $x \sim y$. Then we can find $a \in A, b \in B$ such that $y = axb$. Then $x = a^{-1}yb^{-1}$. Since $a^{-1} \in A$ and $b^{-1} \in B$, we have $y \sim x$. So the relation is symmetric.

Let $x, y, z \in G$ such that $x \sim y$ and $y \sim z$. Then we have $y = axb$ and $z = ayb\beta$ for $a, a \in A$ and $b, \beta \in B$. Hence $z = (aa)x(b\beta)$. Since $aa \in A$ and $b\beta \in B$, we have $x \sim z$. So the relation is transitive.

So this relation is an equivalence relation. Let $x \in G$, let us find its equivalence class $[x]$. By definition:

$$[x] = \{y \in G \mid x \sim y\} = \{y \in G \mid \exists a \in A, b \in B, y = axb\} = \{axb \in G \mid a \in A, b \in B\}$$

which is what we wanted.

**Problem 5.** Let $f : S_n \rightarrow \mathbb{Z}$ be a group homomorphism, and $x \in S_n$. Since $S_n$ is finite, the order $n$ of $x$ is finite. Since $f$ is a homomorphism we have:

$$nf(x) = f(x^n) = f(e) = 0$$

By definition, $n \neq 0$ so $f(x) = 0$. This holds for all $x \in S_n$, so $f$ is the trivial homomorphism.

**Problem 6.** The list of distinct subgroups of $S_3$ is:

- $\{e\}$, which is normal.
- $\langle(12)\rangle, \langle(23)\rangle, \langle(13)\rangle$ which all have order 2 and are not normal.
- $\langle(123)\rangle$, which has order 3 and is normal.
- $S_3$ which is normal.
Problem 7. The list of distinct subgroups of $D_4 = \langle r, s \mid r^4 = s^2 = srs = e \rangle$ is:

- $\{ e \}$, which is normal.
- $\langle r^2 \rangle$, which has order 2 and is normal since $r^2$ is in the center of $D_4$.
- $\langle r \rangle$, which has order 4 and is normal since $srs = r^{-1}$.
- $\langle s \rangle$, $\langle sr \rangle$, $\langle sr^2 \rangle$, $\langle sr^3 \rangle$, which have order 2 and are not normal.
- $\langle s, r^2 \rangle$ and $\langle sr, r^2 \rangle$ which have order 4 and are normal.
- $D_4$ which is normal.

Problem 8. Since $H$ is a subgroup, $e \in H$, so $e = xex^{-1} \in xHx^{-1}$. Given $a, b \in xHx^{-1}$ we can write $a = xhx^{-1}$ and $b = xkx^{-1}$ with $h, k \in H$. Then $ab^{-1} = xhx^{-1}xk^{-1}x^{-1} = x(hk^{-1})x^{-1}$. Since $hk^{-1} \in H$, $ab^{-1} \in xHx^{-1}$. So $xHx^{-1}$ is a subgroup of $G$.

Let us prove that $N = \cap_{x \in G} xHx^{-1}$ is normal. Let $g \in G$. For all $x \in G$ we have:

$$g(xHx^{-1})g^{-1} = (gx)H(gx)^{-1}$$

Hence:

$$gNg^{-1} = \cap_{x \in G} (gx)H(gx)^{-1}.$$ 

The map $G \to G, x \mapsto gx$ is a bijection, so we can make the change of variable $y = gx$ in the intersection, and we get:

$$gNg^{-1} = \cap_{y \in G} yHy^{-1} = N$$

So $N$ is normal.

Problem 9. We have seen previously that $Z(G)$ is a subgroup of $G$. Let us show it is normal. Let $z \in Z(G)$ and $g \in G$. Then we have $gzg^{-1} = zg^{-1}g = z \in Z(G)$. Hence $Z(G)$ is normal.