Math 110AH Fall 18

HOMEWORK 5 - SOLUTIONS

Problem 1. Denote by $\mathcal{S}$ the set of subgroups of G, which is by assumption finite. Consider the map:

$$f : \left\{ \begin{array}{ccc} G & \to & \mathcal{S} \\ g & \mapsto & \langle g \rangle \end{array} \right.$$

If $H \in f(G)$, H is a cyclic group. In particular H has finitely many generators, so $f^{-1}(H)$ is finite. We have:

$$G = \bigsqcup_{H \in \mathcal{S}} f^{-1}(H)$$

So G is a finite union of finite sets, and thus is finite.

Problem 2. Assume $K \cup H$ is a subgroup, and $K \nsubseteq H$. We need to prove that $H \subseteq K$. Fix an element $k \in K \setminus H$. Given $h \in H$, we have $kh \in K \cup H$ since $K \cup H$ is a subgroup. However, $kh \notin H$ (otherwise, $k = khh^{-1} \in H$). Thus $kh \in K$. It follows that $h = k^{-1}kh \in K$. This holds for all $h \in H$, so $H \subseteq K$, which is what we wanted.

Conversely, if $K \subseteq H$ or $H \subseteq K$, we have $K \cup H = H$ or $K \cup H = K$, thus $K \cup H$ is a subgroup.

Problem 3. We start by decomposing H as a union of $H \cap K$ cosets:

$$H = \bigsqcup_{h \in X} h\,(H \cap K)$$

where X denotes a set of representatives of classes in H modulo $H \cap K$. Given $h, h' \in H$, note that:

$$h^{-1}h' \in H \cap K \Leftrightarrow h^{-1}h' \in K$$

So $h\,(H \cap K) = (hK) \cap H$. Since there are finitely many K cosets in G, we deduce that there are finitely many $H \cap K$ cosets in H: X is finite.

We can decompose G as a union of H cosets:

$$G = \bigsqcup_{g \in Y} gH$$

where Y denotes a set of representatives of classes in G modulo H (in particular Y is finite). For all $g \in Y$, we have:

$$gH = \bigsqcup_{h \in X} gh\,(H \cap K)$$

So we deduce:

$$G = \bigsqcup_{(g,h) \in X \times Y} gh\,(H \cap K)$$

Since X and Y are finite, $H \cap K$ has finite index in G.

Problem 4. Let $x \in G$, Since $e \in A, B$ and $x = exe$, we have $x \sim x$. So the relation is reflexive.

Let $x, y \in G$ such that $x \sim y$. Then we can find $a \in A$, $b \in B$ such that $y = axb$. Then $x = a^{-1}yb^{-1}$. Since $a^{-1} \in A$ and $b^{-1} \in B$, we have $y \sim x$. So the relation is symmetric.

Let $x, y, z \in G$ such that $x \sim y$ and $y \sim z$. Then we have $y = axb$ and $z = \alpha y \beta$ for $a, \alpha \in A$ and $b, \beta \in B$. Hence $z = (\alpha a)x(b\beta)$. Since $\alpha a \in A$ and $b\beta \in B$, we have $x \sim z$. So the relation is transitive.

So this relation is an equivalence relation. Let $x \in G$, let us find its equivalence class $[x]$. By definition:

$$\begin{aligned} [x] &= \{y \in G \mid x \sim y\} \\ &= \{y \in G \mid \exists a \in A, b \in B, y = axb\} \\ &= \{axb \in G \mid a \in A, b \in B\} \end{aligned}$$

which is what we wanted.

Problem 5. Let $f : S_n \to \mathbb{Z}$ be a group homomorphism, and $x \in S_n$. Since S_n is finite, the order n of x is finite. Since f is a homomorphism we have:

$$nf(x) = f(x^n) = f(e) = 0$$

By definition, $n \neq 0$ so $f(x) = 0$. This holds for all $x \in S_n$, so f is the trivial homomorphism.

Problem 6. The list of distinct subgroups of S_3 is:

- $\{e\}$, which is normal.
- $\langle(12)\rangle$, $\langle(23)\rangle$, $\langle(13)\rangle$ which all have order 2 and are not normal.
- $\langle(123)\rangle$, which has order 3 and is normal.
- S_3 which is normal

Problem 7. The list of distinct subgroups of $D_4 = \langle r, s \mid r^4 = s^2 = srsr = e\rangle$ is:

- $\{e\}$, which is normal.
- $\langle r^2\rangle$, which has order 2 and is normal since r^2 is in the center of D_4.
- $\langle r\rangle$, which has order 4 and is normal since $srs = r^{-1}$.
- $\langle s\rangle$, $\langle sr\rangle$, $\langle sr^2\rangle$, $\langle sr^3\rangle$, which have order 2 and are not normal.
- $\langle s, r^2\rangle$ and $\langle sr, r^2\rangle$ which have order 4 and are normal
- D_4 which is normal.

Problem 8. Since H is a subgroup, $e \in H$, so $e = xex^{-1} \in xHx^{-1}$. Given $a, b \in xHx^{-1}$ we can write $a = xhx^{-1}$ and $b = xkx^{-1}$ with $h, k \in H$. Then $ab^{-1} = xhx^{-1}xk^{-1}x^{-1} = x(hk^{-1})x^{-1}$. Since $hk^{-1} \in H$, $ab^{-1} \in xHx^{-1}$. So xHx^{-1} is a subgroup of G.

Let us prove that $N = \cap_{x\in G} xHx^{-1}$ is normal. Let $g \in G$. For all $x \in G$ we have:

$$g(xHx^{-1})g^{-1} = (gx)H(gx)^{-1}$$

Hence:

$$gNg^{-1} = \bigcap_{x\in G} (gx)H(gx)^{-1}.$$

The map $G \to G, x \mapsto gx$ is a bijection, so we can make the change of variable $y = gx$ in the intersection, and we get:

$$gNg^{-1} = \bigcap_{y\in G} yHy^{-1} = N$$

So N is normal.

Problem 9. We have seen previously that $Z(G)$ is a subgroup of G. Let us show it is normal. Let $z \in Z(G)$ and $g \in G$. Then we have $gzg^{-1} = zgg^{-1} = z \in Z(G)$. Hence $Z(G)$ is normal.