Hyperbolic Geometric Flow

Kefeng Liu
Zhejiang University
UCLA

Outline

- Introduction
- Hyperbolic geometric flow
- Local existence and nonlinear stability
- Wave character of metrics and curvatures
- Exact solutions and Birkhoff theorem
- Dissipative hyperbolic geometric flow
- Open problems

1. Introduction

- Ricci flow
- Structure of manifolds
- Singularities in manifold and space-time
- Einstein equations and Penrose conjecture
- Wave character of metrics and curvatures
- Applications of hyperbolic PDEs to differential geometry

J. Hong, D. Christodoulou, S. Klainerman, M. Dafermos, I. Rodnianski, H. Lindblad, N. Ziper $\cdots\cdots$

Kong et al (Comm. Math. Phys.; J. Math. Phys. 2006)

2. Hyperbolic Geometric Flow

Let $(\mathscr{M}, g_{ij})$ be n-dimensional complete Riemannian manifold.

The Levi-Civita connection

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left\{ \frac{\partial g_{il}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right\}$$

The Riemannian curvature tensors

$$R^k_{ijl} = \frac{\partial \Gamma^k_{jl}}{\partial x^i} - \frac{\partial \Gamma^k_{il}}{\partial x^j} + \Gamma^k_{ip}\Gamma^p_{jl} - \Gamma^k_{jp}\Gamma^p_{il}, \quad R_{ijkl} = g_{kp} R^p_{ijl}$$

The Ricci tensor

$$R_{ik} = g^{jl} R_{ijkl}$$

The scalar curvature

$$R = g^{ij} R_{ij}$$

Home Page
Title Page
Page 4 of 41
Go Back
Full Screen
Close
Quit

Hyperbolic geometric flow (HGF)

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij} \tag{1}$$

for a family of Riemannian metrics $g_{ij}(t)$ on $\mathscr{M}$.

General version of HGF

$$\frac{\partial^2 g_{ij}}{\partial t^2} + 2R_{ij} + \mathscr{F}_{ij}\left(g, \frac{\partial g}{\partial t}\right) = 0 \tag{2}$$

-**De-Xing Kong and Kefeng Liu:**

Wave Character of Metrics and Hyperbolic Geometric Flow, 2006

Physical background

- **Relation between Einstein equations and HGF**

Consider the Lorentzian metric

$$ds^2 = -dt^2 + g_{ij}(x,t)dx^i dx^j$$

Einstein equations in vacuum, ie., $G_{ij} = 0$ become

$$\frac{\partial^2 g_{ij}}{\partial t^2} + 2R_{ij} + \frac{1}{2} g^{pq} \frac{\partial g_{ij}}{\partial t} \frac{\partial g_{pq}}{\partial t} - g^{pq} \frac{\partial g_{ip}}{\partial t} \frac{\partial g_{kq}}{\partial t} = 0 \qquad (3)$$

This is a special example of general version (2) of HGF. Neglecting the terms of first order gives the HGF (1).

(3) is named as Einstein's hyperbolic geometric flow

- **Applications to cosmology: singularity of universe**

Geometric background

- Structure of manifolds
- Singularities in manifolds
- Wave character of metrics and curvatures
- Long-time behavior and stability of manifolds

Laplace equation, heat equation and wave equation

- Laplace equation (elliptic equations)

$$\triangle u = 0$$

- Heat equation (parabolic equations)

$$u_t - \triangle u = 0$$

- Wave equation (hyperbolic equations)

$$u_{tt} - \triangle u = 0$$

Einstein manifold, Ricci flow, hyperbolic geometric flow

- Einstein manifold (elliptic equations)

$$R_{ij} = \lambda g_{ij}$$

- Ricci flow (parabolic equations)

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$$

- Hyperbolic geometric flow (hyperbolic equations)

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij}$$

Laplace equation, heat equation and wave equation on manifolds in the Ricci sense

Geometric flows

$$\alpha_{ij}\frac{\partial^2 g_{ij}}{\partial t^2}+\beta_{ij}\frac{\partial g_{ij}}{\partial t}+\gamma_{ij}g_{ij}+2R_{ij}=0,$$

where $\alpha_{ij}, \beta_{ij}, \gamma_{ij}$ are certain smooth functions on $\mathscr{M}$ which may depend on t.

In particular,

$\alpha_{ij}=1, \beta_{ij}=\gamma_{ij}=0$: **hyperbolic geometric flow**

$\alpha_{ij}=0, \beta_{ij}=1, \gamma_{ij}=0$: **Ricci flow**

$\alpha_{ij}=0, \beta_{ij}=0, \gamma_{ij}=const.$: **Einstein manifold**

Birkhoff Theorem holds for geometric flows

- Fu-Wen Shu and You-Gen Shen:
  Geometric flows and black holes, arXiv: gr-qc/0610030

All of the known explicit solutions of the Einstein solutions, such as the Schwartzchild solution, Kerr solution, satisfy HGF.

At least for short time solutions, there should be a $1-1$ correspondence between solutions of HGF and the Einstein equation.

Complex geometric flows

If the underlying manifold $\mathscr{M}$ is a complex manifold and the metric is K$\ddot{a}$hler,

$$a_{ij}\frac{\partial^2 g_{i\bar{j}}}{\partial t^2} + b_{ij}\frac{\partial g_{i\bar{j}}}{\partial t} + c_{ij} g_{i\bar{j}} + 2R_{i\bar{j}} = 0,$$

where a_{ij}, b_{ij}, c_{ij} are certain smooth functions on $\mathscr{M}$ which may also depend on t.

3. Local Existence and Nonlinear Stability

Local existence theorem (Dai, Kong and Liu, 2006)

Let $(\mathscr{M}, g^0_{ij}(x))$ be a compact Riemannian manifold. Then there exists a constant $h > 0$ such that the initial value problem

$$\begin{cases} \dfrac{\partial^2 g_{ij}}{\partial t^2}(x,t) = -2R_{ij}(x,t), \\[2ex] g_{ij}(x,0) = g^0_{ij}(x), \ \dfrac{\partial g_{ij}}{\partial t}(x,0) = k^0_{ij}(x), \end{cases}$$

has a unique smooth solution $g_{ij}(x,t)$ on $\mathscr{M} \times [0,h]$, where $k^0_{ij}(x)$ is a symmetric tensor on $\mathscr{M}$.

W. Dai, D. Kong and K. Liu: Hyperbolic geometric flow (I): short-time existence and nonlinear stability

Method of proof

- **Strict hyperbolicity**

Suppose $\hat{g}_{ij}(x,t)$ is a solution of the hyperbolic geometric flow (1), and $\psi_t : \mathscr{M} \to \mathscr{M}$ is a family of diffeomorphisms of $\mathscr{M}$. Let

$$g_{ij}(x,t) = \psi_t^* \hat{g}_{ij}(x,t)$$

be the pull-back metrics. The evolution equations for the metrics $g_{ij}(x,t)$ are strictly hyperbolic.

- **Symmetrization of hyperbolic geometric flow**

Introducing the new unknowns

$$g_{ij},\ h_{ij} = \frac{\partial g_{ij}}{\partial t},\ g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k},$$

we have

$$\begin{cases} \dfrac{\partial g_{ij}}{\partial t} = h_{ij}, \\ g^{kl}\dfrac{\partial g_{ij,k}}{\partial t} = g^{kl}\dfrac{\partial h_{ij}}{\partial x^k}, \\ \dfrac{\partial h_{ij}}{\partial t} = g^{kl}\dfrac{\partial g_{ij,k}}{\partial x^l} + \widetilde{H}_{ij}. \end{cases}$$

Rewrite it as

$$A^0(u)\frac{\partial u}{\partial t} = A^j(u)\frac{\partial u}{\partial x^j} + B(u),$$

where the matrices A^0, A^j are symmetric.

Nonlinear stability

Let $\mathscr{M}$ be a n-dimensional complete Riemannian manifold. Given symmetric tensors g^0_{ij} and g^1_{ij} on $\mathscr{M}$, we consider

$$\begin{cases} \dfrac{\partial^2 g_{ij}}{\partial t^2}(t,x) = -2R_{ij}(t,x) \\ g_{ij}(x,0) = \overline{g}_{ij}(x) + \varepsilon g^0_{ij}(x), \quad \dfrac{\partial g_{ij}}{\partial t}(x,0) = \varepsilon g^1_{ij}(x), \end{cases}$$

where $\varepsilon > 0$ is a small parameter.

Definition The Ricci flat Riemannian metric $\overline{g}_{ij}(x)$ possesses the (locally) nonlinear stability with respect to (g^0_{ij}, g^1_{ij}), if there exists a positive constant $\varepsilon_0 = \varepsilon_0(g^0_{ij}, g^1_{ij})$ such that, for any $\varepsilon \in (0, \varepsilon_0]$, the above initial value problem has a unique (local) smooth solution $g_{ij}(t,x)$;
$\overline{g}_{ij}(x)$ is said to be (locally) nonlinear stable, if it possesses the (locally) nonlinear stability with respect to arbitrary symmetric tensors $g^0_{ij}(x)$ and $g^1_{ij}(x)$ with compact support.

Nonlinear stability theorem (Dai, Kong and Liu, 2006)

The flat metric $g_{ij} = \delta_{ij}$ of the Euclidean space $\mathbb{R}^n$ with $n \geq 5$ is nonlinearly stable.

Remark **The above theorem gives the nonlinear stability of the hyperbolic geometric flow on the Euclidean space with dimension larger than 4. The situation for the 3-, 4-dimensional Euclidean spaces is very different. This is a little similar to the proof of the Poincar$\acute{e}$ conjecture: the proof for the three dimensional case and $n \geq 5$ dimensional case are very different, while the four dimensional smooth case is still open.**

Method of proof

Define a 2-tensor h

$$g_{ij}(x,t) = \delta_{ij} + h_{ij}(x,t).$$

Choose the elliptic coordinates $\{x^i\}$ around the origin in $\mathbb{R}^n$. It suffices to prove that the following Cauchy problem has a unique global smooth solution

$$\begin{cases} \dfrac{\partial^2 h_{ij}}{\partial t^2}(x,t) = \displaystyle\sum_{k=1}^{n} \dfrac{\partial^2 h_{ij}}{\partial x^k \partial x^k} + \bar{H}_{ij}\left(h_{kl}, \dfrac{\partial h_{kl}}{\partial x^p}, \dfrac{\partial^2 h_{kl}}{\partial x^p \partial x^q}\right), \\ h_{ij}(x,0) = \varepsilon g^0_{ij}(x), \quad \dfrac{\partial h_{ij}}{\partial t}(x,0) = \varepsilon g^1_{ij}(x). \end{cases}$$

Einstein's hyperbolic geometric flow

$$\frac{\partial^2 g_{ij}}{\partial t^2} + 2R_{ij} + \frac{1}{2} g^{pq} \frac{\partial g_{ij}}{\partial t} \frac{\partial g_{pq}}{\partial t} - g^{pq} \frac{\partial g_{ip}}{\partial t} \frac{\partial g_{kq}}{\partial t} = 0$$

satisfy the null condition

Global existence and nonlinear stability for small initial data (Dai, Kong and Liu)

Home Page
Title Page
Page 19 of 41
Go Back
Full Screen
Close
Quit

4. Wave Nature of Curvatures

Under the hyperbolic geometric flow (1), the curvature tensors satisfy the following nonlinear wave equations

$$\frac{\partial^2 R_{ijkl}}{\partial t^2} = \Delta R_{ijkl} + \text{(lower order terms)},$$

$$\frac{\partial^2 R_{ij}}{\partial t^2} = \Delta R_{ij} + \text{(lower order terms)},$$

$$\frac{\partial^2 R}{\partial t^2} = \Delta R + \text{(lower order terms)},$$

where Δ is the Laplacian with respect to the evolving metric, the lower order terms only contain lower order derivatives of the curvatures.

Evolution equation for Riemannian curvature tensor

Under the hyperbolic geometric flow (1), the Riemannian curvature tensor R_{ijkl} satisfies the evolution equation

$$\begin{aligned}\frac{\partial^2}{\partial t^2}R_{ijkl} = \;& \triangle R_{ijkl} + 2\left(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}\right)\\ &-g^{pq}\left(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} + R_{ijpl}R_{qk} + R_{ijkp}R_{ql}\right)\\ &+2g_{pq}\left(\frac{\partial}{\partial t}\Gamma^p_{il}\frac{\partial}{\partial t}\Gamma^q_{jk} - \frac{\partial}{\partial t}\Gamma^p_{jl}\frac{\partial}{\partial t}\Gamma^q_{ik}\right),\end{aligned}$$

where $B_{ijkl} = g^{pr}g^{qs}R_{piqj}R_{rksl}$ and $\triangle$ is the Laplacian with respect to the evolving metric.

Home Page
Title Page
◀◀ ▶▶
◀ ▶
Page 21 of 41
Go Back
Full Screen
Close
Quit

Evolution equation for Ricci curvature tensor

Under the hyperbolic geometric flow (1), the Ricci curvature tensor satisfies

$$\begin{aligned}\frac{\partial^2}{\partial t^2}R_{ik} &= \triangle R_{ik} + 2g^{pr}g^{qs}R_{piqk}R_{rs} - 2g^{pq}R_{pi}R_{qk} \\ &\quad +2g^{jl}g_{pq}\left(\frac{\partial}{\partial t}\Gamma^p_{il}\frac{\partial}{\partial t}\Gamma^q_{jk} - \frac{\partial}{\partial t}\Gamma^p_{jl}\frac{\partial}{\partial t}\Gamma^q_{ik}\right) \\ &\quad -2g^{jp}g^{lq}\frac{\partial g_{pq}}{\partial t}\frac{\partial}{\partial t}R_{ijkl} + 2g^{jp}g^{rq}g^{sl}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}R_{ijkl}\end{aligned}$$

Evolution equation for scalar curvature

Under the hyperbolic geometric flow (1), the scalar curvature satisfies

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2} R = \; & \triangle R + 2|\mathrm{Ric}|^2 \\
& + 2 g^{ik} g^{jl} g_{pq} \left(\frac{\partial}{\partial t} \Gamma^p_{il} \frac{\partial}{\partial t} \Gamma^q_{jk} - \frac{\partial}{\partial t} \Gamma^p_{jl} \frac{\partial}{\partial t} \Gamma^q_{ik} \right) \\
& - 2 g^{ik} g^{jp} g^{lq} \frac{\partial g_{pq}}{\partial t} \frac{\partial}{\partial t} R_{ijkl} \\
& - 2 g^{ip} g^{kq} \frac{\partial g_{pq}}{\partial t} \frac{\partial R_{ik}}{\partial t} + 4 R_{ik} g^{ip} g^{rq} g^{sk} \frac{\partial g_{pq}}{\partial t} \frac{\partial g_{rs}}{\partial t}
\end{aligned}
$$

5. Exact Solutions and Birkhoff theorem

5.1 Exact solutions with the Einstein initial metrics

Definition (Einstein metric and manifold) ***A Riemannian metric g_{ij} is called Einstein if $R_{ij} = \lambda g_{ij}$ for some constant λ. A smooth manifold $\mathscr{M}$ with an Einstein metric is called an Einstein manifold.***

If the initial metric $g_{ij}(0, x)$ is Ricci flat, i.e., $R_{ij}(0, x) = 0$, then $g_{ij}(t, x) = g_{ij}(0, x)$ is obviously a solution to the evolution equation (1). Therefore, any Ricci flat metric is a steady solution of the hyperbolic geometric flow (1).

If the initial metric is Einstein, that is, for some constant λ it holds

$$R_{ij}(0,x) = \lambda g_{ij}(0,x), \quad \forall\, x \in \mathscr{M},$$

then the evolving metric under the hyperbolic geometric flow (1) will be steady state, or will expand homothetically for all time, or will shrink in a finite time.

Let

$$g_{ij}(t,x) = \rho(t)g_{ij}(0,x)$$

By the defintion of the Ricci tensor, one obtains

$$R_{ij}(t,x) = R_{ij}(0,x) = \lambda g_{ij}(0,x)$$

Equation (1) becomes

$$\frac{\partial^2(\rho(t)g_{ij}(0,x))}{\partial t^2} = -2\lambda g_{ij}(0,x)$$

This gives an ODE of second order

$$\frac{d^2\rho(t)}{dt^2} = -2\lambda$$

One of the initial conditions is $\rho(0) = 1$, another one is assumed as $\rho'(0) = v$. The solution is given by

$$\rho(t) = -\lambda t^2 + vt + 1$$

Home Page
Title Page
Page 26 of 41
Go Back
Full Screen
Close
Quit

General solution formula is

$$g_{ij}(t,x) = (-\lambda t^2 + vt + c) g_{ij}(0,x)$$

Remark This is different with the Ricci flow!

<u>Case I</u> The initial metric is Ricci flat, i.e., $\lambda = 0$.
In this case,

$$\rho(t) = vt + 1. \tag{4}$$

If $v = 0$, then $g_{ij}(t,x) = g_{ij}(0,x)$. This shows that $g_{ij}(t,x) = g_{ij}(0,x)$ is stationary.

If $v > 0$, then $g_{ij}(t,x) = (1+vt)g_{ij}(0,x)$. This means that the evolving metric $g_{ij}(t,x) = \rho(t)g_{ij}(0,x)$ exists and expands homothetically for all time, and the curvature will fall back to zero like $-\frac{1}{t}$.

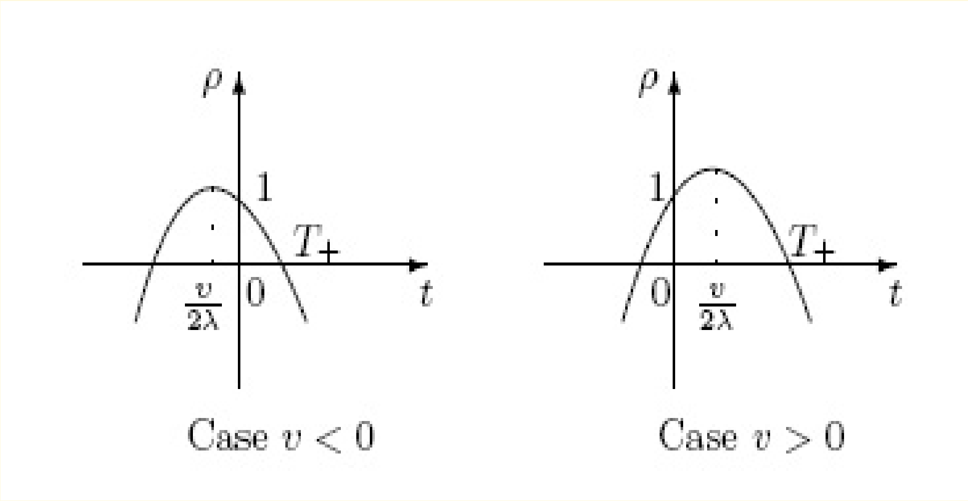
Notice that the evolving metric $g_{ij}(t,x)$ only goes back in time to $-v^{-1}$, when the metric explodes out of a single point in a "big bang".

If $v < 0$, then $g_{ij}(t,x) = (1+vt)g_{ij}(0,x)$. Thus, the evolving metric $g_{ij}(t,x)$ shrinks homothetically to a point as $t \nearrow T_0 = -\frac{1}{v}$. Note that, when $t \nearrow T_0$, the scalar curvature is asymptotic to $\frac{1}{T_0 - t}$. This phenomenon corresponds to the "black hole" in physics.

Conclusion: *For the Ricci flat initial metric, if the initial velocity is zero, then the evolving metric g_{ij} is stationary; if the initial velocity is positive, then the evolving metric g_{ij} exists and expands homothetically for all time; if the intial velocity is negative, then the evolving metric g_{ij} shrinks homothetically to a point in a finite time.*

Case II **The initial metric has positive scalar curvature, i.e., $\lambda > 0$.**

In this case, the evolving metric will shrink (if $v < 0$) or first expands then shrink (if $v > 0$) under the hyperbolic flow by a time-dependent factor.



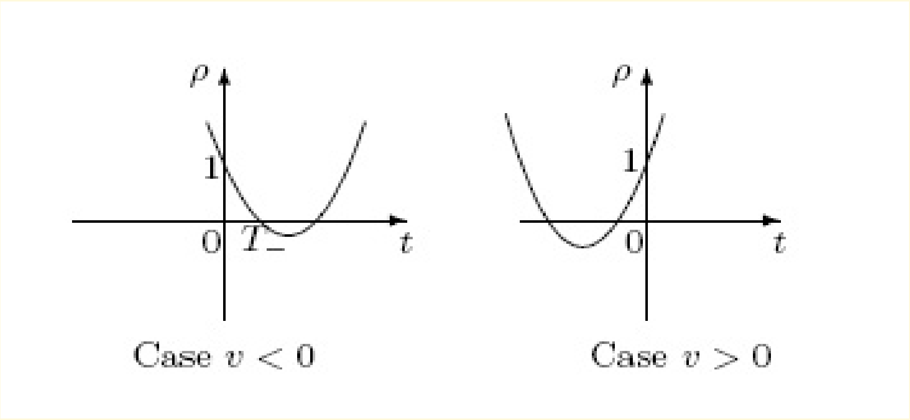
Case III The initial metric has a negative scalar curvature, i.e., $\lambda < 0$.

In this case, we devide into three cases to discuss:

Case 1 $v^2 + 4\lambda > 0$.

(a) $v < 0$: the evolving metric will shrink in a finite time under the hyperbolic flow by a time-dependent factor;

(b) $v > 0$: the evolving metric $g_{ij}(t,x) = \rho(t) g_{ij}(0,x)$ exists and expands homothetically for all time, and the curvature will fall back to zero like $-\frac{1}{t^2}$.



Home Page
Title Page
◀◀ ▶▶
◀ ▶
Page 31 of 41
Go Back
Full Screen
Close
Quit

Case 2 $v^2 + 4\lambda < 0$.

In this case, the evolving metric $g_{ij}(t,x) = \rho(t) g_{ij}(0,x)$ exists and expands homothetically (if $v > 0$) or first shrinks then expands homothetically (if $v < 0$) for all time.

The scalar curvature will fall back to zero like $-\frac{1}{t^2}$.

Case 3 $v^2 + 4\lambda = 0$.

If $v > 0$, then evolving metric $g_{ij}(t,x) = \rho(t) g_{ij}(0,x)$ exists and expands homothetically for all time. In this case the scalar curvature will fall back to zero like $\frac{1}{t^2}$. If $v < 0$, then the evolving metric $g_{ij}(t,x)$ shrinks homothetically to a point as $t \nearrow T_* = \frac{v}{2\lambda} > 0$ and the scalar curvature is asymptotic to $\frac{1}{T_* - t}$.

Remark A typical example of the Einstein metric is

$$ds^2 = \frac{1}{1-\kappa r^2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2,$$

where κ is a constant taking its value -1, 0 or 1. We can prove that

$$ds^2 = R^2(t)\left\{\frac{1}{1-\kappa r^2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right\}$$

is a solution of the hyperbolic geometric flow (1), where

$$R^2(t) = -2\kappa t^2 + c_1 t + c_2$$

in which c_1 and c_2 are two constants. This metric plays an important role in cosmology.

5.2 Exact solutions with axial symmetry

Consider

$$ds^2 = f(t,z)dz^2 - \frac{t}{g(t,z)}\left[(dx - \mu(t,z)dy)^2 + g^2(t,z))dy^2\right],$$

where $f, \; g$ are smooth functions with respect to variables.

Since the coordinates x and y do not appear in the preceding metric formula, the coordinate vector fields ∂_x and ∂_y are Killing vector fields. The flow ∂_x (resp. ∂_y) consists of the coordinate translations that send x to $x + \Delta x$ (resp. y to $y + \Delta y$), leaving the other coordinates fixed. Roughly speaking, these isometries express the x-invariance (resp. y-invariance) of the model. The x-invariance and y-invariance show that the model possesses the z-axial symmetry.

HGF gives

$$g_t = \mu_t = 0$$

$$f = \frac{1}{2g^2}\left[g_z^2 + \mu_z^2\right] + \frac{1}{g^4}\mu_z^2(c_1 t + c_2),$$

where g_z and μ_z satisfy

$$gg_z^2 - gg_z\mu_{zz}\mu_z^{-1} + g_z^2 + \mu_z^2 = 0$$

Birkhoff Theorem holds for axial-symmetric solutions!

Angle speed μ is independent of t!

6. Dissipative hyperbolic geometric flow

Let $\mathscr{M}$ be an n-dimensional complete Riemannian manifold with Riemannian metric g_{ij}. Consider the hyperbolic geometric flow

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij} + 2g^{pq}\frac{\partial g_{ip}}{\partial t}\frac{\partial g_{jq}}{\partial t} + \left(d - 2g^{pq}\frac{\partial g_{pq}}{\partial t}\right)\frac{\partial g_{ij}}{\partial t} + \left(c + \frac{1}{n-1}\left(g^{pq}\frac{\partial g_{pq}}{\partial t}\right)^2 + \frac{1}{n-1}\frac{\partial g^{pq}}{\partial t}\frac{\partial g_{pq}}{\partial t}\right)g_{ij}$$

for a family of Riemannian metrics $g_{ij}(t)$ on $\mathscr{M}$, where c and d are arbitrary constants.

By calculations, we obtain the following evolution equation of the scalar curvature R with respect to the metric $g_{ij}(x,t)$

$$
\begin{aligned}
\frac{\partial^2 R}{\partial t^2} = \; & \triangle R + 2|Ric|^2 + \left(d - 2g^{pq}\frac{\partial g_{pq}}{\partial t}\right)\frac{\partial R}{\partial t} - \\
& \left(c + \frac{1}{n-1}\left(g^{pq}\frac{\partial g_{pq}}{\partial t}\right)^2 + \frac{1}{n-1}\frac{\partial g^{pq}}{\partial t}\frac{\partial g_{pq}}{\partial t}\right)R + \\
& 2g^{ik}g^{jl}g_{pq}\frac{\partial \Gamma^p_{ij}}{\partial t}\frac{\partial \Gamma^q_{kl}}{\partial t} - 2g^{ik}g^{jl}g_{pq}\frac{\partial \Gamma^p_{ik}}{\partial t}\frac{\partial \Gamma^q_{jl}}{\partial t} + \\
& 8g^{ik}\frac{\partial \Gamma^q_{ip}}{\partial t}\frac{\partial \Gamma^p_{kq}}{\partial t} - 8g^{ik}\frac{\partial \Gamma^p_{ip}}{\partial t}\frac{\partial \Gamma^q_{kq}}{\partial t} - 8g^{ik}\frac{\partial \Gamma^q_{pq}}{\partial t}\frac{\partial \Gamma^p_{ik}}{\partial t}
\end{aligned}
$$

Introduce

$$y \triangleq g^{pq}\frac{\partial g_{pq}}{\partial t} = Tr\left\{\frac{\partial g_{pq}}{\partial t}\right\}$$

and

$$z \triangleq g^{pq}g^{rs}\frac{\partial g_{pr}}{\partial t}\frac{\partial g_{qs}}{\partial t} = \left|\frac{\partial g_{pq}}{\partial t}\right|^2.$$

By (1), we have

$$\frac{\partial y}{\partial t} = -2R - \frac{n-2}{n-1}y^2 + dy - \frac{1}{n-1}z + cn$$

Global stability of Euclidean metric

- Dai, Kong and Liu, 2006

7. Open Problems

♦ **Yau has several conjectures about asymptotically flat manifolds with nonnegative scalar curvature. HGF supplies a promising way to approach these conjectures.**

♦ **Penrose cosmic censorship conjecture. Given initial metric g^0_{ij} and symmetric tensor k_{ij}, study the singularity of the HGF with these initial data.**

These problems are from general relativity and Einstein equation in which HGF has root.

♦ **Global existence and singularity; HGF has global solution for small initial data.**

♦ **HGF flow and (minimal) hypersurface. Study HGF with initial data given by initial metric and second fundamental form h_{ij}.**

There was an approach of geometrization by using Einstein equation which is too complicated to use. HGF may simplify and even complete the approach.

Thank you