On $SL_2(\mathbf{Z})$ and Topology

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In this note we describe some new topological results derived from the modular invariance of certain elliptic operators on loop spaces under the action of $SL_2(\mathbf{Z})$. The results include very general rigidity, divisibility and vanishing theorems in topology.

1 Loop Groups and Rigidity Theorems

Let M be a smooth compact spin manifold with an S^1-action and P be an elliptic operator on M commuting with the action. Then both the kernal and the cokernal of P are S^1-modules. The Lefschetz number, or the character-valued index of P at $g \in S^1$ is defined to be

$$F_P(g) = \mathrm{tr}_g \mathrm{Ker} P - \mathrm{tr}_g \mathrm{Coker} P \in R(S^1)$$

where $R(S^1)$ is the character ring of the S^1-modules. We say that P is rigid with respect to this S^1-action, if $F_P(g)$ is independent of g.

Two well-known examples of rigid elliptic operators are the signature operator d_s and the Dirac operator D [AH].

Now let $\tilde{L}\mathrm{Spin}(2l)$ denote the central extension of the loop group $L\mathrm{Spin}(2l)$ and E be a positive energy representation of it. Then under the rotation action of the loop, E has decomposition $\oplus_{n\geq 0}^{\infty} E_n$ where E_n's are finite dimensional representations of $\mathrm{Spin}(2l)$. Given a real rank $2l$ spin vector bundle V on M, let P be its frame bundle and $\tilde{E}_n$ be the bundle associated to P and E_n. We define

$$\psi(E, V) = \sum_{n\geq 0}^{\infty} \tilde{E}_n q^n \in K(M)[[q]]$$

where $q = e^{2\pi i\tau}$ with τ in the upper half plane is a parameter.

Assume that there exists an S^1-action on M which lifts to V. Let $p(\cdot)_{S^1}$ denote the first equivariant Pontrjagin class. Then we have the following:

Theorem 1 *For every positive energy representation E of $\tilde{L}Spin(2l)$ of highest weight of level m, if $p_1(M)_{S^1} = mp_1(V)_{S^1}$, then*

$$D \otimes \otimes_{n=1}^{\infty} S_{q^n}(TM) \otimes \psi(E, V)$$

is rigid.

Here $S_t(\cdot)$ is the symmetric operation in $K(M)[[t]]$. Theorem 1 actually holds for any simply connected Lie group, instead of Spin$(2l)$.

2 Jacobi Forms and Vanishing Theorems

We consider the equivariant cohomology group of M, $H^*_{S^1}(M, \mathbf{Z})$. We know that $H^*_{S^1}(M, \mathbf{Z})$ is a module over $H^*(BS^1, \mathbf{Z})$ induced by the projection

$$\pi : \quad M \times_{S^1} ES^1 \to BS^1.$$

We are interested in the situation when $p_1(V)_{S^1} - p_1(M)_{S^1} \in H^*_{S^1}(M, \mathbf{Z})$ is equal to the pull-back of an element in $H^*(BS^1, \mathbf{Z})$. Since $H^*(BS^1, \mathbf{Z}) = \mathbf{Z}[[u]]$ with u a generator of degree 2, we know that this is equivalent to

$$p_1(V)_{S^1} - p_1(M)_{S^1} = n \cdot \pi^* u^2$$

with n an integer. We call n the anomaly to rigidity.

Let M and V be as above. Let $\Lambda_t(\cdot)$ be the wedge operation in $K(M)[[t]]$ and

$$\begin{aligned}
\Theta'_q(TM|V)_v &= \otimes_{n=1}^{\infty} \Lambda_{q^n}(V - \dim V) \otimes_{m=1}^{\infty} S_{q^m}(TM - \dim M), \\
\Theta_q(TM|V)_v &= \otimes_{n=1}^{\infty} \Lambda_{-q^{n-\frac{1}{2}}}(V - \dim V) \otimes_{m=1}^{\infty} S_{q^m}(TM - \dim M), \\
\Theta_{-q}(TM|V)_v &= \otimes_{n=1}^{\infty} \Lambda_{q^{n-\frac{1}{2}}}(V - \dim V) \otimes_{m=1}^{\infty} S_{q^m}(TM - \dim M), \\
\Theta^*_q(TM|V)_v &= \otimes_{n=1}^{\infty} \Lambda_{-q^n}(V - \dim V) \otimes_{m=1}^{\infty} S_{q^m}(TM - \dim M).
\end{aligned}$$

Denote the spinor bundles of V by $\triangle(V) = \triangle^+(V) \oplus \triangle^-(V)$. We then have

Theorem 2 *Assume*

$$p_1(V)_{S^1} - p_1(M)_{S^1} = n \cdot \pi^* u^2,$$

then the Lefschetz numbers of $D \otimes \triangle(V) \otimes \Theta_q'(TM|V)_v, D \otimes \Theta_q(TM|V)_v$, $D \otimes \Theta_{-q}(TM|V)_v$ *and* $D \otimes (\triangle^+(V) - \triangle^-(V)) \otimes \Theta_q^*(TM|V)_v$ *are holomorphic Jacobi forms of index* $\frac{n}{2}$ *and weight* $k = \frac{1}{2} dim M$ *over* $(2\mathbf{Z})^2 \rtimes \Gamma$ *with* Γ *equal to* $\Gamma_0(2)$, $\Gamma^0(2)$, Γ_θ *and* $SL_2(\mathbf{Z})$ *respectively.*

Here by Lefschetz number we actually mean its extension from unit circle to complex plane. See [Sh] for definitions of the modular subgroups appeared above.

The following elliptic operator

$$D \otimes \otimes_{m=1}^{\infty} S_{q^m}(TM - \mathrm{dim} M)$$

corresponds to the Dirac operator on LM. we have the following $\hat{\mathfrak{A}}$-vanishing theorem for loop space.

Theorem 3 *If* $p_1(M)_{S^1} = n \cdot \pi^* u^2$ *for some integer* n*, then the Lefschetz number, especially the index of*

$$D \otimes \otimes_{m=1}^{\infty} S_{q^m}(TM - dim M)$$

is zero.

We note that $p_1(M)_{S^1} = n \cdot \pi^* u^2$ is the equivariant spin condition on LM. If M is 2-connected, this condition is equivalent to $p_1(M) = 0$. Theorem 2 can be generalized to higher level case.

Theorem 4 *Let* M*,* V *and* E *be as in Theorem 1. If*

$$m p_1(V)_{S^1} - p_1(M)_{S^1} = n \cdot \pi^* u^2,$$

then

$$q^{m_\Lambda} D \otimes \otimes_{n=1}^{\infty} S_{q^n}(TM - dim M) \otimes \psi(E, V)$$

is a holomorphic Jacobi form of index $\frac{nm}{2}$ *and weight* k *over* $(2\mathbf{Z})^2 \rtimes \Gamma(N(m))$*.*

Here $N(m)$ is a positive integer, m_Λ is a rational number depending on the level and the weight of E. There are similar theorems for almost complex manifolds.

3 Corollaries and Examples

In this section we give several corollaries of our theorems. We will see that Theorem 1 actually holds in much more general situations. Let M and V be as in Theorem 1. Consider the case of level $m = 1$. Then $\tilde{L}\mathrm{Spin}(2l)$ has four irreducible positive energy representations which we denote by S^+, S^-, S'_+ and S'_-. Then take E to be $S = S^+ + S^-$, $T = S^+ - S^-$, $S' = S'_+ + S'_-$ or $T' = S'_+ - S'_-$, one gets the rigidity of the following four elliptic operators

$$D \otimes \triangle(V) \otimes_{n=1}^{\infty} \Lambda_{q^n}(V) \otimes_{m=1}^{\infty} S_{q^m}(TM),$$

$$D \otimes (\triangle^+(V) - \triangle^-(V)) \otimes_{n=1}^{\infty} \Lambda_{-q^n}(V) \otimes_{m=1}^{\infty} S_{q^m}(TM),$$

$$D \otimes \otimes_{n=1}^{\infty} \Lambda_{-q^{n-\frac{1}{2}}}(V) \otimes_{m=1}^{\infty} S_{q^m}(TM),$$

$$D \otimes \otimes_{n=1}^{\infty} \Lambda_{q^{n-\frac{1}{2}}}(V) \otimes_{m=1}^{\infty} S_{q^m}(TM).$$

Corollary 1 (Witten Rigidity Theorem) *The above four elliptic operators are rigid under the conditions $p_1(V)_{S^1} = p_1(M)_{S^1}$ and $w_2(V) = w_2(M) = 0$.*

Take four non-negative integers a, b, c, d and consider the representation

$$Q_{a,b,c,d} = S^{\otimes a} \otimes S'^{\otimes b} \otimes T^{\otimes c} \otimes T'^{\otimes d}.$$

¿From Theorem 1 we have

Corollary 2 *If $p_1(M)_{S^1} = (a+b+c+d)p_1(V)_{S^1}$ and $w_2(M) = w_2(V) = 0$, then*

$$D \otimes \otimes_{m=1}^{\infty} S_{q^m}(TM) \otimes \psi(Q_{a,b,c,d}, V)$$

is rigid.

So $\{S, S', T, T'\}$ generate a graded ring by tensor product, each homogeneous term of degree m gives a rigid elliptic operator, if the $\mathrm{Spin}(2l)$-bundle V satisfies $p_1(M)_{S^1} = mp_1(V)_{S^1}$.

If we have another $\mathrm{Spin}(2n)$-vector bundle W such that

$$ap_1(V)_{S^1} + bp_1(W)_{S^1} = p_1(M)_{S^1}$$

for some non-negative integers a, b, then we have that, for two highest weight positive energy representations E and F of level a and b of $\tilde{L}\mathrm{Spin}(2l)$ and $\tilde{L}\mathrm{Spin}(2n)$ respectively, the operator

$$D \otimes \otimes_{m=1}^{\infty} S_{q^m}(TM) \otimes \psi(E, V) \otimes \psi(F, W)$$

is rigid. One can get much more examples in this way.

One corollary of Theorem 2 is the following

Corollary 3 *Let M, V and n be as in Theorem 2. If $n = 0$, the Lefschetz numbers of the elliptic operators in Theorem 2 are independent of the generators of S^1. If $n < 0$ or $n = 2$ and $k{=}\frac{1}{2}dimM$ is odd, then their Lefschetz numbers are identically zero, especially their indices are zero.*

¿From Theorem 4 we can get similar results. If $n < 0$, the Lefschetz number of the elliptic operator in Theorem 4 must be zero, so is its index. Especially this includs the generalization of Corollary 2 to the non-zero anomaly case. If $n = 0$, Theorem 4 reduces to Theorem 1.

One can draw more corollaries from our theorems. For example we have that, if $p_1(M)_{S^1} - mp_1(V)_{S^1} = n \cdot \pi^* u^2$ with the integer $n \leq 0$, then $D \otimes V$ and $D \otimes \triangle(V) \otimes V$ are rigid; if $n < 0$, then their indices vanish.

Under certain condition all of our results have analogues for almost complex manifolds. For example let X be a compact almost complex manifold of complex dimension k and W be a complex vector bundle of rank l on X. Here by complex bundle we mean a real bundle with complex structure. One has the decompositions

$$TX \otimes \mathbf{C} = T'X \oplus T''X, \ W \otimes \mathbf{C} = W' \oplus W''.$$

Assume that there exists an S^1-action on X which lifts to W and preserves the complex structures of X and W. Let $L = \mathrm{det}W'$, $K = \mathrm{det}T'X$ and

$$\Theta_q^\alpha(TX|W) = \otimes_{n=0}^\infty \Lambda_{-y^{-1}q^n} W'' \otimes_{n=1}^\infty \Lambda_{-yq^n} W' \otimes_{n=1}^\infty S_{q^n} T'X \otimes_{n=1}^\infty S_{q^n} T''X.$$

We then have

Theorem 5 *Assume $w_2(W) = w_2(X)$, $c_1(W) \equiv 0(modN)$ and $p_1(W)_{S^1} = p_1(X)_{S^1}$, then the elliptic operator $\bar\partial \otimes (K^{-1} \otimes L)^{\frac{1}{2}} \otimes \Theta_q^\alpha(TX|W)$ is rigid.*

Take $W = TX$, we get the rigidity theorem of Hirzebruch [H]. If $p_1(W)_{S^1} - p_1(X)_{S^1} = n \cdot \pi^* u^2$, we get a holomorphic Jacobi form. If $n < 0$, we have vanishing theorem.

Fix two positive integers m, ν with $\nu < 2m$. Let $\{l_j\}$ be integers such that $m_j \nu = 2ml_j + k_j$ with $k_j \geq 0$ [BT]. Similarly let $\{p_j\}$ be given by $n_j \nu = 2mp_j + q_j$ with $q_j \geq 0$. Let $\{M_{2m}^i\}$ be the fixed submanifolds of the subgroup $\mathbf{Z}_{2m}$ of S^1 and D_m^i be the Dirac operator on M_{2m}^i. Let V_m^i denote the $\mathbf{Z}_{2m}$ invariant part of V restricted to M_{2m}^i and $e(\cdot)$ denote Euler number. Combining with the rigidity theorems, we study the behaviors of $F^V_{d_s}(t,\tau)$ and $F^V_{D^*}(t,\tau)$ around the singular fibers of some elliptic modular surfaces and get the following localization formulas:

Theorem 6

$$Ind(D\otimes\triangle(V))=\sum_{M^i_{2m}}(-1)^{\sum_j l_j}Ind(D^i_m\otimes\triangle(V^i_m)),$$

$$e(V)=\sum_{M^i_{2m}}(-1)^{\sum_j l_j-\sum_j p_j}e(V^i_m).$$

We have similar results for almost complex manifolds. The study also reveals the connection between the transfer argument of [BT], [H] and [Kr] and the singular fibers of certain elliptic modular surfaces.

4 Ideas of Proofs

We first sketch the proof of Theorem 1. To display our idea clearly, we restrict our attention to the isolated fixed point case. First since $\tilde{L}\mathrm{Spin}(2l)$ is simply connected, we can assume that E is a level m integrable highest weight module $L(\Lambda)$ of the affine Lie algebra $\hat{L}so(2l)$.

Let $\{p\}\subset M$ be the fixed points of a generator $g=e^{2\pi it}\in S^1$. Let $\{m_j\}$ and $\{n_\nu\}$ be the exponents of TM and V respectively, at the fixed point p. That is, we have orientation-compatible decompositions

$$TM|_p=E_1\oplus\cdots\oplus E_k,\ k=\frac{1}{2}\mathrm{dim}M,$$

$$V|_p=L_1\oplus\cdots\oplus L_l,\ l=\frac{1}{2}\mathrm{dim}V$$

where $\{E_j\},\{L_\nu\}$ are complex line bundles and g acts on E_j and L_ν by $e^{2\pi i m_j t}$ and $e^{2\pi i n_\nu t}$ respectively.

Let $\theta(v,\tau),\theta_1(v,\tau),\theta_2(v,\tau),\theta_3(v,\tau)$ be the four Jacobi theta-functions [Ch]. Consider the following functions

$$H(t,\tau)=(2\pi i)^{-k}\prod_{j=1}^{k}\frac{\theta'(0,\tau)}{\theta(m_jt,\tau)},$$

$$c_E(t,\tau)=\chi_E(T,\tau)$$

where $T = (n_1 t, \cdots, n_l t)$ and $\chi_E(z,\tau) = q^{m_\Lambda}\mathrm{ch}_E(z,\tau)$ is the normalized Kac-Weyl character of the representation $E = L(\Lambda)$ of $\tilde{L}\mathrm{Spin}(2l)$ [Ka].

First we find that

$$F_E(t,\tau) = \sum_p H(t,\tau)c_E(t,\tau)$$

is the Lefschetz number of

$$q^{m_\Lambda} \cdot D \otimes \otimes_{n=1}^{\infty} S_{q^n}(TM - \mathrm{dim}M) \otimes \psi(E, V).$$

Obviously we can extend $F_E(t,\tau)$ to a meromorphic function on $\mathbf{C} \times \mathbf{H}$. The rigidity theorem is therefore equivalent to the proof that $F_E(t,\tau)$ is independent of t. Together with a trick about modular transformations, the proof consists of the following three lemmas.

Lemma 1 *If $p_1(M)_{S^1} = mp_1(V)_{S^1}$, then $F_E(t,\tau) = \sum_p H(t,\tau)c_E(t,\tau)$ is invariant under the action*

$$t \to t + a\tau + b$$

for $a, b \in 2\mathbf{Z}$.

Recall that the modular transformation of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$ on $(t,\tau) \in \mathbf{C} \times \mathbf{H}$ is given by

$$g(t,\tau) = (\frac{t}{c\tau + d}, \frac{a\tau + b}{c\tau + d}).$$

Lemma 2 *For any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$, we have*

$$F_E(g(t,\tau)) = (c\tau + d)^k F_{gE}(t,\tau)$$

where $gE = \sum_\mu a_\mu E_\mu$ is a finite $\mathbf{C}$-linear combination of positive energy representations of $\tilde{L}Spin(2l)$ of highest weight of level m.

These two lemmas are proved by using the transformation formulas of theta-functions, the theorem of Kac-Peterson about the modularity of the characters of the integrable highest weight modules of affine Lie algebras. The condition on Pontrjagin classes is used very crucially, but there is no need to consider any parity of the exponents.

By generalizing an observation of [BT] we have

Lemma 3 *For any $g \in SL_2(\mathbf{Z})$, the function $F_{gE}(t,\tau)$ is holomorphic in (t,τ) for $t \in \mathbf{R}$ and $\tau \in \mathbf{H}$.*

Now we can *prove Theorem 1*. By Lemma 1, we know that $F_E(t,\tau)$ is a doubly periodic meromorphic function in t, therefore to get the rigidity theorem, we only need to prove that $F_E(t,\tau)$ is holomorphic on $\mathbf{C} \times \mathbf{H}$.

First note that, as a meromorphic function on $\mathbf{C} \times \mathbf{H}$, all of the possible polar divisors of $F_E(t,\tau)$ can be expressed in the form $t = \frac{n(c\tau+d)}{A}$ with A, n, c, d integers, $A \neq 0$ and c, d prime to each other. We find integers a, b such that $ad - bc = 1$ and consider the matrix $g = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in SL_2(\mathbf{Z})$.

Since
$$F_{gE}(t,\tau) = (-c\tau + a)^{-k} F_E\left(\frac{t}{-c\tau + a}, \frac{d\tau - b}{-c\tau + a}\right),$$
it is easy to see that, if $t = \frac{n(c\tau+d)}{A}$ is the polar divisor of $F_E(t,\tau)$, then a polar divisor of $F_{gE}(t,\tau)$ is given by

$$\frac{t}{-c\tau + a} = \frac{n(c\frac{d\tau+b}{-c\tau+a} + d)}{A}$$

which exactly gives $t = \frac{n}{A}$. This is a contradiction to Lemmas 2 and 3. So $F_E(t,\tau)$ is holomorphic on $\mathbf{C} \times \mathbf{H}$. This proves Theorem 1 for the isolated fixed point case. The general fixed point case can be discussed in the same manner. See [Liu4] for the details.

For the proof of Theorem 2, let us denote the Lefschetz numbers of $2^{-l} \cdot D \otimes \triangle(V) \otimes \Theta'_q(TM|V)_v, D \otimes \Theta_q(TM|V)_v,\ D \otimes \Theta_{-q}(TM|V)_v$ and $D \otimes (\triangle^+(V) - \triangle^-(V)) \otimes \Theta^*_q(TM|V)_v$ by $F^V_{d_s}(t,\tau)$, $F^V_D(t,\tau)$, $F^V_{-D}(t,\tau)$ and $F^V_{D^*}(t,\tau)$ respectively. Apply the Lefschetz fixed point formula, we have

$$\begin{aligned}
F^V_{d_s}(t,\tau) &= (2\pi i)^{-k} \sum_p \frac{\theta'(0,\tau)^k}{\theta_1(0,\tau)^l} \frac{\prod_{\nu=1}^l \theta_1(n_\nu t,\tau)}{\prod_{j=1}^k \theta(m_j t,\tau)}, \\
F^V_D(t,\tau) &= (2\pi i)^{-k} \sum_p \frac{\theta'(0,\tau)^k}{\theta_2(0,\tau)^l} \frac{\prod_{\nu=1}^l \theta_2(n_\nu t,\tau)}{\prod_{j=1}^k \theta(m_j t,\tau)}, \\
F^V_{-D}(t,\tau) &= (2\pi i)^{-k} \sum_p \frac{\theta'(0,\tau)^k}{\theta_3(0,\tau)^l} \frac{\prod_{\nu=1}^l \theta_3(n_\nu t,\tau)}{\prod_{j=1}^k \theta(m_j t,\tau)}, \\
F^V_{D^*}(t,\tau) &= (2\pi i)^{-k} \sum_p \theta'(0,\tau)^{k-l} \frac{\prod_{\nu=1}^l \theta(n_\nu t,\tau)}{\prod_{j=1}^k \theta(m_j t,\tau)}.
\end{aligned}$$

Similarly the Lefschetz number $H(t,\tau)$ of

$$D \otimes \otimes_{m=1}^{\infty} S_{q^m}(TM - \dim M)$$

is given by

$$H(t,\tau) = (2\pi i)^{-k} \sum_p \prod_{j=1}^{k} \frac{\theta'(0,\tau)}{\theta(m_j t,\tau)}.$$

We can similarly extend these F^V's and H to well-defined meromorphic functions on $(t,\tau) \in \mathbf{C} \times \mathbf{H}$. The key point is to prove that they are holomorphic which is accomplished by using the basic transformation formulas of theta-functions. The proof of the vanishing theorems uses the rather elementary fact that there is no holomorphic Jacobi form of negative index except 0. We refer the reader to [Liu4] for the detail.

5 Elliptic Genera and Elliptic Functions

Theta-function gives the best way to characterize the three universal elliptic genera. For convenience we change variable $u = \pi v$ in the Jacobi theta-functions and still write them as $\theta(u,\tau)$, $\theta_1(u,\tau)$, $\theta_2(u,\tau)$ and $\theta_3(u,\tau)$.

Let Ω be the lattice generated by $(\pi, \pi\tau)$, Ω_1 by $(\pi, 2\pi\tau)$, Ω_2 by $(2\pi, \pi\tau)$ and Ω_3 by $(\pi - \pi\tau, \pi + \pi\tau)$. Let $\mathfrak{P}(\mathfrak{u})$ be the Weirstrass $\mathfrak{P}$-function associated to the lattice 2Ω. We start from the Weirstrass parametrization of the elliptic curve

$$\mathfrak{P}'(u)^2 = 4(\mathfrak{P}(\mathfrak{u}) - \mathfrak{e}_1)(\mathfrak{P}(\mathfrak{u}) - \mathfrak{e}_2)(\mathfrak{P}(\mathfrak{u}) - \mathfrak{e}_3).$$

By looking at their poles and zeroes, one finds that $\mathfrak{P}(\mathfrak{u}) - \mathfrak{e}_\mathfrak{j}$ for $j = 1,2,3$ have well-defined square roots on the whole u-plane. Define $f_j(u)$ such that

$$f_j(u)^2 = \mathfrak{P}(\mathfrak{u}) - \mathfrak{e}_\mathfrak{j}.$$

Then each $f_j(u)$ is an elliptic function with period lattice $2\Omega_j$. We call $f_1(u)$, $f_2(u)$, $f_3(u)$ the root functions.

We denote the characteristic series of $d_s \otimes \Theta_q'(TM|TM)_v$, $D \otimes \Theta_q(TM|TM)_v$ and $D \otimes \Theta_{-q}(TM|TM)_v$ by $f_{d_s}(x)$, $f_D(x)$ and $f_{-D}(x)$ respectively. See Section 2. The indices of these operators are the so-called universal elliptic genera. We have

$$f_{d_s}(u) = \frac{1}{2i} \frac{\theta_1(u,\tau)\theta'(0,\tau)}{\theta(u,\tau)\theta_1(0,\tau)},$$

$$f_D(u) = \frac{1}{2i}\frac{\theta_2(u,\tau)\theta'(0,\tau)}{\theta(u,\tau)\theta_2(0,\tau)},$$

$$f_{-D}(u) = \frac{1}{2i}\frac{\theta_3(u,\tau)\theta'(0,\tau)}{\theta(u,\tau)\theta_3(0,\tau)}.$$

We also have

$$f_1(u) = 2if_{d_s}(u),\ f_2(u) = 2if_D(u),\ f_3(u) = 2if_{-D}(u).$$

¿From these we can easily derive all of the basic properties of elliptic genera. For example we have their functonal equations:

$$f'_{d_s}(u)^2 = (f_{d_s}(u)^2 - \frac{1}{4}\theta_3^4)(f_{d_s}(u)^2 - \frac{1}{4}\theta_2^4),$$

$$f'_D(u)^2 = (f_D(u)^2 + \frac{1}{4}\theta_3^4)(f_D(u)^2 + \frac{1}{4}\theta_1^4),$$

$$f'_{-D}(u)^2 = (f_{-D}(u)^2 + \frac{1}{4}\theta_2^4)(f_{-D}(u)^2 - \frac{1}{4}\theta_1^4).$$

Here we use the notation $\theta_j = \theta_j(0,\tau)$. Compare with the standard equation of elliptic genus

$$y^2 = 1 - 2\delta x^2 + \varepsilon x^4$$

we get the following formulas:

$$\text{For } d_s \otimes \Theta'_q(TM|TM)_v:\quad \delta' = \frac{1}{8}(\theta_2^4 + \theta_3^4),\ \varepsilon' = \frac{1}{16}\theta_2^4\theta_3^4;$$

$$\text{For } D \otimes \Theta_q(TM|TM)_v:\quad \delta = -\frac{1}{8}(\theta_1^4 + \theta_3^4),\ \varepsilon = \frac{1}{16}\theta_1^4\theta_3^4;$$

$$\text{For } D \otimes \Theta_{-q}(TM|TM)_v:\quad \delta_- = \frac{1}{8}(\theta_1^4 - \theta_2^4),\ \varepsilon_- = -\frac{1}{16}\theta_1^4\theta_2^4.$$

There are more formulas relating the universal elliptic genera to elliptic functions. See [Liu4].

6 Divisibility and Miraculous Cancellations

In the non-equivariant situation, we can also get some results by using $SL_2(\mathbf{Z})$.

Theorem 7 *Let M be a dimension $8k+4$ compact smooth spin manifold and V be a rank $2l$ real spin vector bundle.*

a) If $p_1(V)=0$, then Ind $d_s\otimes\triangle(V)\equiv 0(mod 2^{l+3})$.

b) If $p_1(V)=p_1(M)$ and $l\geq 4k+2$, then Ind$D\otimes\triangle(V)\equiv 0(mod 16)$.

Take V to be a trivial bundle in a) or $V=TM$ in b), one recovers the result of Ochanine: sign$(M)\equiv 0(\mathrm{mod}16)$. We can also generalize this theorem to the higher level case.

We only describe the idea of the proof of b), which we owe to Hirzebruch. Consider the modular transformation between the two elliptic operators

$$D\otimes\triangle(V)\otimes\Theta_q'(TM|V)_v \text{ and } D\otimes\Theta_q(TM|V)_v.$$

First using theta-functions we show that their indices are modular forms of weight $4k+2$ over $\Gamma_0(2)$ and $\Gamma^0(2)$ respectively. We also know that $4\delta', 16\varepsilon'$ are generators of the ring of modular forms of intergral Fourier coefficients over $\Gamma_0(2)$; similarly $8\delta, \varepsilon$ are the generators over $\Gamma^0(2)$. The weight of δ, δ' is 2 and the weight of $\varepsilon, \varepsilon'$ is 4. Therefore we can write

$$\mathrm{Ind}D\otimes\Theta_q(TM|V)=a_0(8\delta)^{2k+1}+a_1(8\delta)^{2k-1}\varepsilon+\cdots+a_k(8\delta)\varepsilon^k$$

where the a_j's are integral linear combinations of the indices of the Dirac operator twisted by the Fourier coefficients of $\Theta_q(TM|V)$. For example

$$\begin{aligned} a_0 &= -\mathrm{Ind}D; \\ a_1 &= \mathrm{Ind}D\otimes V+(24(2k+1)-2l)\mathrm{Ind}D. \end{aligned}$$

Apply the modular transformation $S:\ \tau\to-\frac{1}{\tau}$, we get

$$\begin{aligned} \mathrm{Ind}D\otimes\triangle(V)\otimes\Theta_q'(TM|V)_v &= 2^l[a_0(8\delta')^{2k+1}+a_1(8\delta')^{2k-1}\varepsilon'+\cdots+a_k(8\delta')\varepsilon'^k] \\ &\equiv 0(\mathrm{mod}16). \end{aligned}$$

Let V be as in b) of Theorem 7. One easy consequence of the proof is the following formula which was considered to involve the miraculous cancellation of [AW].

Corollary 4 *Ind $D\otimes\triangle(V)=2^{l+2k+1}\cdot\sum_{j=0}^k 2^{-6j}a_j$.*

One can get similar type of results for the operator in a) and for complex manifolds with $c_1\equiv 0(\mathrm{mod}N)$.

It is also interesting to use modular forms to study the Dirac operator on LM, from which, if $p_1(M)=0$, we get a modular form of weight $\frac{1}{2}\mathrm{dim}M$

over $SL_2(\mathbf{Z})$. This gives us some strong estimates on $\mathrm{Ind} D \otimes S^m(TM)$ when $m \to \infty$.

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