RELATIONS AMONG FIXED POINTS

KEFENG LIU

Let M be a smooth manifold with a circle action, and $\{P\}$ be the fixed point sets. The problem I want to discuss in this paper is how to get the topological information of one relatively complicated fixed point set, say P_0, from the other much simpler fixed points. Such problems are interesting in symplectic geometry and geometric invariant theory, especially in the study of moduli spaces. In this paper I derive several very simple integral formulas which express integrals over P_0 in terms of integrals over the other fixed point sets P's. As applications, I use these formulas to give an explicit expression for integrations of cohomology classes on the moduli space of higher rank stable bundles over a Riemann surface in terms of integrals over lower rank moduli spaces. In rank 2 case, these formulas express the integrals over the moduli spaces in terms of integrals over symmetric products of the Riemann surface.

These formulas are also useful in computing the changes of integrals on the quotient manifolds when the polarization is altered in geometric invariant theory [DH], [Th], or when the level of moment map is changed in sympletic geometry [GS]. On the other hand, recently Pidstrigach and Tyurin [PT] have constructed a circle action on the moduli space of solutions of a rank 2 Seiberg-Witten equation whose fixed point sets are respectively given by the moduli spaces of self-dual connections and a rank 1 Seiberg-Witten equation. This indicates that our formulas may be useful in relating the Donaldson invariants to the Seiberg-Witten invariants.

In §5 we also derive a similar formula in equivariant K-theory which relates the theorem in [GS1] about geometric quantization commuting with symplectic reduction to the Verlinde formula. Note that for rank 2 case, the formulas we derived can be used to recover many reasults about moduli spaces of vector bundles on a Riemann surface, such as the Verlinde formula and the Newstead conjectures about the vanishing of Chern classes and Pontryagin classes.

By using the same idea we can derive similar formulas on noncompact manifolds, such as the Hitchin moduli spaces of vector bundles with Higgs fields on a compact Riemann surface. We note that, when the rank and degree of the vector bundle are prime to each other, the Hitchin moduli spaces are smooth complete symplectic manifolds. In §6 we write down some localization formulas for Hitchin's moduli spaces and indicate their proofs.

In this note, I restrict myself to the derivation of the theoretical formulas. In a forthcoming paper I will carry out some actual computations.

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1. Integral formulas

We first assume the manifold M is compact and without boundary. Let V be the Killing vector field generated by the circle action, let $d_t = d - ti_V$ where i_V is the contraction operator and t is a complex parameter, be the differential for equivariant cohomology $H^*_{S^1}(M)$. Consider an even degree equivariant cohomology class which can be written in the form

$$\eta = \eta_{2k} + t\eta_{2k-2} + \cdots + t^k\eta_0$$

where η_j is a smooth j-form on M with the relations

$$d\eta_{j-2} = i_V\eta_j, \quad j = 2, 4, \cdots, 2k.$$

We call $2k$ the degree of η and denote by $\deg\eta$.

Let B_P be a small invariant neighborhood around P and $U_P = \partial B_P/S^1$. The normal bundle ν_P of P in M is decomposed into sum of complex line bundles according to the circle action:

$$\nu_P = L_1 \oplus \cdots \oplus L_l$$

where $2l$ is the codimension of P in M. The generator g of S^1 acts on L_j by g^{m_j}. It is easy to see that U_P is a fiber bundle over P. The fiber is weighted projective space $P(m_1, \cdots, m_l)$. Recall that the equivariant Euler characteristic class of ν_P is given by

$$e_t(\nu_P) = \prod_{j=1}^{l}(c_1(L_j) + m_jt)$$

where $c_1(L_j)$ is the first Chern class of L_j. Let $i : \partial B_P \to M$ be the inclusion map. Since

$$H^*_{S^1}(\partial B_P) \cong H^*(U_P),$$

we have the standard homomorphism

$$\rho_P : \; H^*_{S^1}(M) \to H^*_{S^1}(\partial B_P) \cong H^*(U_P)$$

induced by the inclusion i. We will use i_P to denote the inclusion of P in M and i^*_P denote the pull-back map in cohomology.

Now let P_0 be one of the fixed point set. Take out an invariant small neighborhood B_0 of P_0. Let $U_0 = \partial B_0/S^1$ and ρ_{P_0} be the homomorphism from $H_{S^1}(M)$ to $H^*(U_0)$. Let

$$\eta = \eta_{2k} + t\eta_{2k-2} + \cdots + t^k\eta_0$$

be equivariant with $\deg \eta < n$ where n is the dimension of M. Then our basic formulas are

Formula 1:

$$\int_{U_0} \rho_{P_0}(\eta) = -\sum_{P \neq P_0} \mathrm{res}_{t=0} \int_P \frac{i^*_P\eta}{e_t(\nu_P)}$$

where the $\mathrm{res}_{t=0}$ means taking the coefficient of t^{-1}, and

Formula 2:

$$\int_{U_0} \rho_{P_0}(\eta) = \sum_{P \neq P_0} \int_{U_P} \rho_P(\eta).$$

These two formulas will be our basic tools to study the relations among fixed points. They are very useful in the following situation:

Let E be an equivariant vector bundle on M, then it descends to U_0 and U_P by quotient, the equivariant characteristic classes of E, say $c_r(E)_t$, is mapped by ρ_{P_0} and ρ_P to the non-equivariant characteristic classes of the corresponding quotient bundles E_{P_0}, E_P over U_0, U_P correspondingly. That is, we have

$$\rho_{P_0}(c_r(E)_t) = c_r(E_{P_0}) \text{ and } \rho_P(c_r(E)_t) = c_r(E_P)$$

respectively.

To *prove* the above formulas, we first note that if $\deg\, \eta < n$, then from the localization formula [AB], we get

$$\sum_P \int_P \frac{i^*_P\eta}{e_t(\nu_P)} = 0.$$

Next we look at each term on the right hand side. Fix an S^1-invariant metric g on M and consider B_P. Note that there is no fixed point on the bounadry ∂B_P, so

$$\partial B_P \to U_P = \partial B_P/S^1$$

is a circle bundle.

Lemma: *Assume* $\deg \eta <$ n*, where* n *is the dimension of* M*, then*

$$\int_{U_P} \rho_P(\eta) = -\text{res}_{t=0} \int_P \frac{i_P^*\eta}{e_t(\nu_P)}$$

Proof: The proof is easy. Let

$$\theta = g(V,\cdot)/g(V,V), \ \ \beta = \theta/d_t\theta$$

be the Bott forms, then θ and β are well defined outside P. Note that $d_t\beta = 1$. Choose a small ε neiborhood $B_\varepsilon(P)$ of P inside B_P and integrate η over B_P. Apply Stokes theorem, since $\deg \eta < n$, $\int_{B_P} \eta = 0$, we have

$$0 = \int_{B_P} \eta = \lim_{\varepsilon\to 0} \int_{B_P - \ B_\varepsilon(P)} d_t(\eta\beta)$$
$$= -\lim_{\varepsilon\to 0} \int_{\partial B_\varepsilon(P)} \eta\beta + \int_{\partial B_P} \eta\beta.$$

The limit of the first term in the last sum was derived by Bott:

$$-\int_P \frac{i_P^*\eta}{e_t(\nu_P)}.$$

For the second term, note that θ is actually a connection form for the principal bundle $\partial B_P \to U_P$, so $d_t\theta = F - t$ is the equivariant curvature and the integral is given by

$$\int_{\partial B_P} \eta\theta/(F-t) = \int_{U_P} \rho_P(\eta)/(F-t).$$

Since

$$1/(F-t) = -t^{-1}(1 + Ft^{-1} + F\Lambda F t^{-2} + \cdots),$$

take the coefficients of t^{-1} of both sides, we get the lemma. □

Formula 1 is obtained by applying the Lemma to P_0, while Formula 2 to each the of the fixed point sets P's.

Obviously the above discussion equally applies to manifolds with boundary. Let ∂M be the boundary. Assume the action has no fixed point on ∂M and let $M_0 = \partial M/S^1$. Let

$$\rho : \quad H^*_{S^1}(M) \to H^*(M_0)$$

be the induced map by the inclusion of $\partial M \subset M$ and the isomorphism

$$H^*_{S^1}(\partial M) \cong H^*(M_0).$$

Let $\eta \in H^*_{S^1}(M)$ be an element of degree less than the dimension of M. We then have

$$\int_{U_0} \rho_0(\eta) = \sum_{P \neq P_0} \int_{U_P} \rho_P(\eta) + \int_{M_0} \rho(\eta).$$

For an equivariant vector bundle E as above, let E_0 be the descended bundle on M_0, then $\rho(c_r(E)_t) = c_r(E_0)$. In this case the above formula takes a much simpler form

Formula 3:

$$\int_{U_0} c_r(E_{P_0}) = \sum_{P \neq P_0} \int_{U_P} c_r(E_P) + \int_{M_0} c_r(E_0).$$

In the above formulas we always take equivariant differential forms of degree less than the dimension of M. Note that by this we do not lose any topological information of the fixed point sets.

It is easy to generalize the above formulas to manifolds with quotient singularities. One simply replaces the integrals by the corresponding integrals over V-manifolds.

2. Simple examples.

We first look at some straightforward applications of the integral formulas in §1.

A. Let S^1 act on $\mathbf{C^n}$ by

$$e^{it}(z_1, \cdots, z_n) = (e^{im_1 t} z_1, \cdots, e^{im_n t} z_n).$$

The only fixed point is the origin 0. Consider a small ball B around the origin, then the quotient $\partial B/S^1$ is a weighted projective space $P(m_1, \cdots, m_n)$.

For any differential form $\eta = \eta_{2k} + t\eta_{2k-2} + \cdots + t^k \eta_0$ on $\mathbf{C^n}$ with $d_t \eta = 0$ and degree less than $2n$, let $\rho(\eta)$ be its image in $H^*(P(m_1, \cdots, m_n))$, then cut out a small neighborhood of the origin and apply the above formulas to get

$$\int_{P(m_1,\cdots,m_n)} \rho(\eta) = -\text{res}_{t=0} \frac{\eta_0(0)}{t^{n-k}\prod_{j=1}^n m_j}$$

which is zero except when $k = n-1$. It is easy to see that every element in $H^*(P(m_1, \cdots, m_n))$ is in the image of ρ.

B. Consider the following S^1-action on CP^n:

$$[z_0, \cdots, z_n] \to [z_0, \cdots, z_{n-1}, e^{imt}z_n]$$

where $e^{it} \in S^1$ is the generator and m is an integer. The fixed point sets are $P = [0, \cdots, 0, 1]$ and CP^{n-1}. Let $\eta = \eta_{2k} + t\eta_{2k-2} + \cdots + t^k\eta_0$ on $\mathbf{CP^n}$ with $d_t\eta = 0$ and degree less than $2n$. Note that for a small neiborhood B of CP^{n-1}, ∂B is a circle bundle and $\partial B/S^1 = CP^{n-1}$. Let $\rho(\eta)$ be the image of η in $H^*(CP^{n-1})$, we then get

$$\int_{CP^{n-1}} \rho(\eta) = -\text{res}_{t=0} \frac{t^k\eta_0(P)}{(-mt)^n}$$

which is 0 if $k \neq n-1$ and is $(-1)^{n+1}\eta_0(P)/m^n$ if $k = n-1$.

C. Let M be a compact smooth symplectic manifold of dimension $2n$, with a Hamiltonian circle action. Let

$$\mu : \ M \to \mathbf{R}$$

be the moment map. For two points $a < b \in \mathbf{R}$, let $\mu^{-1}(a),\ \mu^{-1}(a)$ be their level sets. Assume the S^1 acts on these two level sets freely and let $M_a,\ M_b$ be their corresponding symplectic quotients. Let $\rho_a,\ \rho_b$ be the homomorphisms as defined in last section. Then it easily follows from Formula 3 that, for any $\eta \in H_{S^1}(M)$ with deg $\eta < 2n$

$$\int_{M_b} \rho_b(\eta) - \int_{M_a} \rho_a(\eta) = -\sum_P \text{res}_{t=0} \int_P \frac{i_P^*\eta}{e_t(\nu_P)}$$

$$= \sum_P \int_{U_P} \rho_P(\eta)$$

where $\{P\}$ are the fixed point sets lying between the two level sets $\mu^{-1}(a)$ and $\mu^{-1}(b)$.

¿From [GS] we know that M_a and M_b are related by a series of flips. The above formulas tell us how the integrals on the two symplectic quotients are related to each other. From [DH] and [Th] we know that, when the polarizations change, the geometric invariant theory quotient also changes by flips. From the abelian model in [Th], we know that the above formula applies to this situation. Note that each U_P in the above formula is still symplectic manifold with induced symplectic form from that of M.

One interesting case is to take $\eta = \omega_t^{n-1} = (\omega + t\mu)^{n-1}$ and to apply the above formula. Let ω_P be the induced symplectic form on U_P, we have

$$\int_{M_b} \omega_b^{n-1} - \int_{M_a} \omega_a^{n-1} = -\sum_P \operatorname{res}_{t=0} \int_P \frac{i_P^* {\omega_t}^{n-1}}{e_t(\nu_P)} = \sum_P \int_{U_P} \omega_P^{n-1}$$

where $\omega_a,\ \omega_b$ are the induced symplectic forms on M_a and M_b respectively. This gives us a formnula expressing the change of symplectic volume for the symplectic quotients at different levels of the moment map.

3. Moduli spaces: rank 2

We first review the construction in [BDW]. Let S be a compact Riemann surface and E a rank 2 complex vector bundle on it with fixed determinant L. A holomorphic structure on E will be denoted by $\bar\partial_E$. For some Hermitian metric H, let us consider the τ-vortex equation introduced by Bradlow

$$\sqrt{-1}\Lambda F_{\bar\partial_E,H} + \frac{1}{2}\phi \otimes \phi^* = \frac{\tau}{2} I.$$

Here $F_{\bar\partial_E,H}$ is the curvature of the torsion free connection of H, and $\Lambda F_{\bar\partial_E,H}$is the contraction against the Kahler form on S. Also ϕ is a holomorphic section with respect to $\bar\partial_E$, ϕ^* its Hermitian adjoint; $\phi \otimes \phi^*$ is considered as a section of $\Omega^0(\operatorname{End} E)$ and I is the identity section.

It is proved by Bradlow that this equation has solution iff τ is in the following *admissible* range

$$\frac{d}{2} \le \tau \le d$$

where $d > 4g - 4$ is the degree of L. A value τ is called generic if it is not an integer in $[d/2, d]$. A pair $(\bar\partial_E, \phi)$ for which the equation has a solution H will be called a (semi-)stable pair. The gauge group acts on a pair by

$$g(\bar\partial_E, \phi) = (g\bar\partial_E g^{-1}, g\phi)$$

In [BDW], it is proved that the moduli space of the solutions for *some* τ to the above vortex equation is a compact Hausdorff topological manifold $\mathcal{B}$ with a circle action

$$e^{it}(\bar{\partial}_E, \phi) = (\bar{\partial}_E, e^{it}\phi).$$

The fixed point sets consists of pairs with the following properties:

1) When $\phi = 0$. This is the moduli space of rank 2 semistable bundles $\mathcal{M}(2, L)$ with fixed determinant L.

2) When E splits into sum of line bundles $F \oplus L \otimes F^{-1}$ where F is uniquely determined by its section ϕ. In fact the zero points of ϕ is a divisor of degree $d - j$ which is just F. This gives us the symmetric product of the Riemann surface, $S^{(d-j)}$ for j an integer lying in $(d/2, d)$, consisting of the divisors of ϕ which are considered as points in S, as the fixed points.

The moment map of the circle action is given by $f: \mathcal{B} \to \mathcal{R}$ such that

$$f(\bar{\partial}_E, \phi) = \frac{1}{8\pi}||\phi||^2_{L^2} + \frac{d}{2}.$$

Its image precisely lies in $\tau \in [d/2, d]$. All sigularities of $\mathcal{B}$ in this rank 2 case lie in $\mathcal{M}(2, L)$. The circle action on the level set of a generic value τ in the image is free with symplectic quotient

$$\mathcal{B}_\tau = f^{-1}(\tau)/S^1$$

which is a Kahler manifold.

We can choose an invariant neighborhood B around $\mathcal{M}(2, L)$ such that both $\mathcal{B} - B$ and ∂B are smooth. Let $U_\mathcal{M} = \partial B/S^1$ then $U_\mathcal{M}$ is a projective bundle over the smooth part of $\mathcal{M}$. By applying our formulas to $\mathcal{B}$, we get, for deg $\eta < d + 2g - 2 = \dim \mathcal{B}$,

$$\int_{U_\mathcal{M}} \rho_0(\eta) = -\sum_j \mathrm{res}_{t=0} \int_{S^{(d-j)}} \frac{i^*\eta}{e_t(\nu_j)}$$

where ν_j is the normal bundle of $S^{(d-j)}$ in $\mathcal{B}$. Also

$$\rho_0: H^*_{S^1}(\mathcal{B}) \to H^*(U_\mathcal{M})$$

is the map as introduced in §1.

The ν_j's in the above formula have the following explicit descriptions. Each of them is a direct sum of two vector bundles on $S^{(d-j)}$:

$$\nu_j = W_j^+ \oplus W_j^-$$

where the circle acts on W^+ by multiplications of e^{2it}, on W^- by e^{-2it}. More explicitly, on $S^{(d-j)}$ the universal pair (V, ϕ) on $\mathcal{B}$ splits into direct sum $\mathcal{O}(\mathcal{D}) \oplus \mathcal{L}(-\mathcal{D})$ with D a divisor of degree $d - j$. The circle action on $\mathcal{O}(\mathcal{D})$ is given by multiplication by e^{it}. Let $\pi : S \times S^{(d-j)} \to S^{(d-j)}$

be the projection and still let D denote the universal divisor, then from [BDW], [Th1] we have

$$W_j^- = (R^0\pi)\mathcal{O}_D L(-D),\ W_j^+ = (R^1\pi)L^{-1}(2D).$$

If we assume the degree of E, $d > 4g-4$ is odd, let E be a universal bundle over $S\times\mathcal{M}(2,L)$, then $U_{\mathcal{M}}$ is just the projective bundle $P(\pi_! E)$ where π is the projection

$$\pi :\quad S\times\mathcal{M}(2,L) \to \mathcal{M}(2,L).$$

It is easy to see that V, when reduced to $U_{\mathcal{M}}$, is $\pi_1^* E\otimes\mathcal{O}(1)$ where $\pi_1 :\ S\times P(\pi_! E) \to S\times\mathcal{M}(2,L)$ is the projection and $\mathcal{O}(1)$ is the anti-tautological line bundle on $P(\pi_! E)$.

Let $v = c_1(\mathcal{O}(1))$, then any element in $H^*(U_{\mathcal{M}})$ can be written in the form

$$b = b_0 + b_1 v + \cdots + b_m v^m$$

where m is the dimension of the fiber and b_j's are cohomology classes in $H^*(\mathcal{M}(2,L))$. We then have

$$\int_{U_{\mathcal{M}}} b = \int_{\mathcal{M}(2,L)} b_m$$

which reduces integrals on $U_{\mathcal{M}}$ to integrals on $\mathcal{M}(2,L)$.

Since the cohomology classes of $\mathcal{M}(2,L)$ are given by the Chern classes of universal bundles, we can take the combinations of the equivariant characteristic classes of $p_!(V^*\otimes V)$ and $p_! V$ where

$$p :\ S\times\mathcal{B}\to\mathcal{B}$$

is the natural projection, to get all possible characteristic classes of $U_{\mathcal{M}}$. By using Formula 3, we get an integral formula on $\mathcal{M}$ in terms of the integrals on the symmetric products of the Riemann surface. While restricted to the symmetric products, the vector bundle splits into sums of line bundles. In this sense we can say that our formula 'abelianizes' the integrals on $\mathcal{M}$. For example take

$$\eta = c_1(p_!(V^*\otimes V))_t^n c_1(p_! V)_t^m$$

where $n = 3g-3$ and $n+m = \dim\mathcal{B} - 1$, we get the volume of $\mathcal{B}$. Note that

$$\rho_0(c_1(p_! V)_t) = c_1(\pi_! E) + v,\ \rho_0(c_1(p_!(V\otimes V)_t) = -c_1(\mathcal{T})$$

where $\mathcal{T}$ denotes the tangent bundle of $\mathcal{M}(2,L)$. When restricted to $S^{(d-j)}$ these classes can be easily computed as in §7 of [Th1].

In the even degree case, we still have a surjective homomorphism from $U_{\mathcal{M}}$ to $\mathcal{M}$, which is a projective bundle over the smooth part. By deleting the sigularities which will not change the integral, we can still use the above method to reduce the integrals on $U_{\mathcal{M}}$ to integrals on $\mathcal{M}(2, L)$.

Let $U_j = P(W_j^+ \oplus W_j^-)$ be the weighted projective bundle on $S^{(d-j)}$as described above and

$$\rho_j : \ H^*_{S^1}(\mathcal{B}) \to H^*(U_j)$$

be the natural map. An equivalent integral formula from Formula 2 is given by

$$\int_{U_{\mathcal{M}}} \rho_0(\eta) = \sum_j \int_{U_j} \rho_j(\eta)$$

which may be more interesting for computations in algebraic geometry. In fact for any generic value τ, we have an integral formula for the symplectic quotient $\mathcal{B}_\tau$:

$$\int_{\mathcal{B}_\tau} \rho_\tau(\eta) = -\sum_{j>\tau} \mathrm{res}_{t=0} \int_{S^{(d-j)}} \frac{i^*\eta}{e_t(\nu_j)}$$

where ρ_τ is the natrual map from the equivariant cohomology of $\mathcal{B}$ to the nonequivariant cohomology of $\mathcal{B}_\tau$.

4. Moduli spaces: higher rank

Now consider the semistable pairs for a holomorphic bundle E of rank $r > 2$, with fixed determinant L and degree $d > r(2g-2)$. It is proved in [BDW] that there is a compact Hausdorff topological space $\mathcal{B}$ parametrizing the solutions of the vertex equation for *some* τ. There is also an open set $\mathcal{B}_0 \subset \mathcal{B}$ which has a natural Kahler V-manifold structure. These spaces have the following properties

1) There is an S^1-action on $\mathcal{B}_0$ which is holomophic and symplectic. The corresponding moment map $f : \mathcal{B}_0 \to \mathbf{R}$ and the action extends continuously to $\mathcal{B}$.

2) At regular levels, the symplectic quotients are smooth Kahler manifolds. The singular points of $\mathcal{B}$ all lie in the critical levels of f.

3) The fixed point sets of the S^1-action on $\mathcal{B}$ are moduli spaces of isomorphism calsses of semistable pairs. Especially one extreme is $\mathcal{M}(r, L)$, the moduli space of holomorphic bundles with rank r and determinant L; while another extreme is $\mathcal{M}(r-1.L)$.

The extended moment map

$$f : \ \mathcal{B} \to R$$

is given by

$$f((E,\phi)) = \frac{||\phi||^2}{4\pi r} + \frac{d}{r}.$$

The critical levels of f, which we denote by I, are the rational numbers $\tau = p/q$ in $[d/r, d/(r-1)]$ with $0 < q < r$. Let $\{\mathcal{Z}_\tau\}_{\tau\in I}$ denote the fixed point sets of the circle action other than $\mathcal{M}(r,L)$. Then $\mathcal{Z}_\tau$ consists of the moduli spaces of semi-stable pairs of lower rank. More precisely, over $\mathcal{Z}_\tau$ the pair (E,ϕ) splits into direct sum $(E_\phi,\phi)\oplus\oplus_j E_j$ where (E_ϕ,ϕ) is a semistable pair and the E_j's are stable bundles each of which has slope τ.

The $\mathcal{Z}_\tau$'s may be singular, but our integral Formula 2 still applies. In fact we can first choose a neiborhood D_τ around each $\mathcal{Z}_\tau$ such that on the boundary of D_τ, the S^1-action is free and

$$U_\tau = \partial D_\tau / S^1$$

has V-manifold structure. Apply the argument in §1 to $\mathcal{B} - \cup_{\tau\in I} D_\tau$ and an equivariant cohomology class η with $\deg\eta < \dim\mathcal{B}$, we get

$$\int_{U_\mathcal{M}} \rho_0(\eta) = \sum_{\tau\in I}\int_{U_\tau}\rho_\tau(\eta).$$

Here as in the rank 2 case, $U_\mathcal{M}$ is still a projective bundle over the smooth part of $\mathcal{M}(r,L)$. Especially if $(d,r)=1$, let E be the unviversal bundle on $S\times\mathcal{M}(r,L)$ and

$$\pi:\ S\times\mathcal{M}(r,L)\to\mathcal{M}(r,L)$$

be the projection, then $U_\mathcal{M} = P(\pi_! E)$. In general we have a surjective homomorphism from $U_\mathcal{M}$ to $\mathcal{M}(r,L)$.

The U_τ's can also be explicitly written as the weighted projective bundle $P(W_\tau^+\oplus W_\tau^-)$ over $\mathcal{Z}_\tau$, where $W_\tau^\pm$ are the stable and unstable bundle of $\mathcal{Z}_\tau$ correspondingly. They can be explicitly described by the filtration of (E,ϕ) [BDW].

We then can use the same method as in the rank 2 case to get integrals of cohomology classes on $\mathcal{M}(r,L)$. For example assume $(r,d)=1$, and let V be the universal bundle on the smooth part of $\mathcal{B}$ and p be the natural projection

$$p:\ S\times\mathcal{B}\to\mathcal{B},$$

then take

$$\eta = c_1(p_!(V^*\otimes V))_t^n c_1(p_! V)_t^m$$

with $n = \dim\mathcal{M}(r,L)$ and $m+n=\dim\mathcal{B}-1$ gives an expression of the volume integral of $\mathcal{M}(r,L)$ in terms of integrals in the neiborhoods of lower rank moduli spaces.

Let $\pi^\tau: U_\tau \to Z_\tau$ be the natural projection, then this formula can be written as

$$\int_{U_\mathcal{M}} \rho_0(\eta) = \sum_{\tau\in I} \int_{Z_\tau} \pi_*^\tau(\rho_\tau(\eta))$$

where π_*^τ denotes the push-forward in cohomology. To use this formula for practical computations, we have to study the singularity of $\mathcal{B}$ more carefully.

5. A K-theory formula

In this section we explain a simple observation about the relation between the multiplicity formula of Guillemin-Sternberg [GS1] and the Verlinde formula as discussed in [Th1].

Let $\mathcal{B}$ be the moduli space given in §3, and $K_{S^1}(\mathcal{B})$ be its equivariant K-group of complex vector bundles. Let

$$\mathrm{Ind}_{S^1}: K_{S^1}(\mathcal{B}) \to K_{S^1}(pt)$$

be the equivariant index map which compute the equivariant Riemann-Roch numbers, the index of $\bar\partial \otimes E$ of a holomorphic vector bundles E on $\mathcal{B}$. Then for any holomorphic equivariant vector bundle L on $\mathcal{B}$, we have the following localization formula in K theory due to Atiyah-Segal:

$$\mathrm{Ind}_{S^1} L = \sum_P \mathrm{Ind}^P(\frac{i^*L}{\Lambda_t\nu})$$

where the sum is over the fixed point sets $\{P\}$, i^* is the restriction map, $\Lambda_t\nu$ denotes the K-theory equivariant Euler class of the normal bundle of the fixed point set P in $\mathcal{B}$, and Ind^P is the non-equivariant index map on P.

Now let L be the determinant line bundle of the universal bundle $\mathcal{V}$ on $S\times\mathcal{B}$ under the map

$$p: S\times\mathcal{B} \to \mathcal{B}$$

and k be a positive integer, we have

$$\mathrm{Ind}_{S^1} L^k = \sum_j a_j z^{m_j} \in K_{S^1}(pt)$$

with $z = e^{it}$. By [GS1], we know that the multiplicity a_j is equal to $\mathrm{Ind}^j L_j^k$ where Ind^j is the nonequivariant index map on $\mathcal{B}_j = f_k^{-1}(m_j)/S^1$

and L_j is the reduced line bundle on $\mathcal{B}_j$ from L. Here $f_k = kf$ with f the moment map given in §3.

Let $r = kd/2$ for k even and $r = [kd/2] + 1$ for k and d odd, we then have

$$a_r = \mathrm{res}_{z=0} z^{-r-1} \mathrm{Ind}_{S^1} L^k = \mathrm{Ind}^0 L_0^k$$

where $L_0 \to U_{\mathcal{M}}$ is the reduced line bundle and Ind^0 is the index map on $U_{\mathcal{M}}$. By combining with the above Atiyah-Segal localization formula, we get

$$\mathrm{Ind}^0 L_0^k = \mathrm{res}_{z=0} z^{-r-1} \sum_{j=0} \mathrm{Ind}^j \left(\frac{i^* L^k}{\Lambda_t \nu_j}\right)$$

where the sum is over the symmetric products $S^{(d-j)}$ of the Riemann surface S and Ind^j is the index map on $S^{(d-j)}$. There is no contribution from $\mathcal{M}$ to the residue which can be easily seen from the fact that the exponent of the circle action on the normal bundle of $\mathcal{M}$ in $\mathcal{B}$ is $z = e^{it}$ and the action on $i^* L \to \mathcal{M}$ is z^{r-1}. In fact the contribution from $\mathcal{M}$ is

$$\mathrm{res}_{z=0} z^{-r-1} \int_{\mathcal{M}} \mathrm{Td}(T\mathcal{M}) \frac{z^{r-1} \mathrm{ch} i^* L}{\prod_{j=1}^{s} (1 - z^{-1} e^{-x_j})}$$

where $\{x_j\}$ are the Chern roots of the normal bundle of $\mathcal{M}$ in $\mathcal{B}$, $s = d - g + 1$ is the codimension of $\mathcal{M}$ and $Td(T\mathcal{M})$ is the Todd class of $\mathcal{M}$. It is easy to see that when d is larger than $2g - 2$ with $g > 1$, this expression is zero.

Recall that $\nu_j = W_j^+ \oplus W_j^-$ as in §3 and by definition

$$\Lambda_z \nu_j = \Lambda_z W_j^+ \otimes \Lambda_{z^{-1}} W_j^-$$

where

$$\Lambda_z W_j^+ = 1 + z W_j^+ + z^2 \Lambda^2 W_j^+ + \cdots$$

and $\Lambda_{z^{-1}} W_j^-$ is defined in the same way.

The left hand side of the above formula gives us the dimension predicted by the Verlinde formula [Th1], while the right hand side is a sum over symmetric products of the Riemann surface whose computation has been completely carried out in §7 of [Th1]. This simple observation implies that in this special case the algebro-geometric computations in [Th1] are basically equivalent to the result in [GS1]: geometric quantization commutes with symplectic reduction.

Note that it is possible that $U_{\mathcal{M}}$ is actually $\mathcal{M}$, but the above discussion still works. The reason is that $\mathcal{M}$ is smooth and the Guillemin-Sternberg multiplicity formula still holds in this case.

6. Localization on Hitchin moduli

We still let X be a compact Riemann surface of genus g. Let V be a holomorphic vector bundle of rank n and degree d on X with $(n\,,\,d)=1$. Let Φ be a smooth section of $\operatorname{End} V \otimes K$ where K is the canonical bundle of X.

Let A be a connection on V, Consider the moduli space $\mathcal{H}$ of solutions of the following Higgs equation introduced by Hitchin [H],

$$F_A = -[\Phi, \Phi^*], \quad \bar{\partial}_A \Phi = 0.$$

It is proved by Hitchin that $\mathcal{H}$ is a smooth, noncompact, complete hyperkahler manifold whose dimension is twice of the dimension of $\mathcal{M}$, the moduli space of stable bundles of rank n and degree d. For example, in the $n=2$ case, $\dim \mathcal{H} = 6(g-1)$. Here the completeness is with respect to the natural Weil-Peterson metric on $\mathcal{M}$.

There exists a holomorphic circle action on $\mathcal{H}$ given by

$$(A, \Phi) \to (A, e^{i\theta}\Phi)$$

which is semifree and has only finitely many fixed point components. The corresponding moment map is

$$\mu(A, \Phi) = \frac{1}{2}||\Phi||_{L^2}^2 = i\int_X \operatorname{Tr}(\Phi\Phi^*)$$

which is a proper map to R. It is easy to check that $d\mu = -i_Y\omega$ where Y is the vector field generated by the circle action and ω denotes the Weil-Peterson Kahler form on $\mathcal{H}$.

The fixed point sets of this action are as follows:

1). $\Phi = 0$, this is the moduli space of stable bundles of rank n and degree d.

2). $\Phi \neq 0$, these are some kinds of lower rank moduli spaces.

a) $n=2$, all of them are 2^{2g}-fold covering of the symmetric product of X.

b) $n=3$, rank 2 vortex moduli spaces and symmetric products of X.

c) $n>3$, not very clear.

Simpson has constructed Higgs moduli spaces for higher dimensional projective manifolds. It will be interesting to study similar localization formulas for the Higgs moduli of an algebraic surface. In the following we briefly discuss the generalization of the foumulas in §1 to this noncompact setting.

1. Equivariant cohomology localization. For simplicity we only consider the symplectic volume. We still denote the moduli space of degree d and rank n stable bundles on X by $\mathcal{M}$.

Let $\tilde{\omega} = \omega + t\mu$ be the equivariant symplectic form. We have $(d - ti_Y)\tilde{\omega} = 0$. Let $\{P_j\}$ be the lower rank moduli sapces which are the fixed point sets of the circle action on $\mathcal{H}$. Let ω_0 be the Kahler form which is the restriction of ω. We then have the following integral formula

$$\int_{\mathcal{M}} e^{\omega_0} = -\mathrm{res}_{t=0} t^{s-1} \sum_{P_j} \int_{P_j} \frac{i^*_{P_j}(e^{\tilde{\omega}})}{e_t(\nu_j)}$$

where $i^*_{P_j}$ is the restriction map, s is the codimension of $\mathcal{M}$ and $e_t(\nu_j)$ is the equivariant Euler class of the normal bundle ν_j of P_j in $\mathcal{H}$.

Remark: a). The equivariant Euler classes $e_t(\nu_j)$ are very clear in this case [H].

b). Formulas involving more general equivariant differential forms can be derived similarly.

2. Equivariant K-theory localization. Let L be the determinant line bundle on $\mathcal{H}$, its restriction to $\mathcal{M}$ gives L_0. Both L and L_0 are ample line bundles. Let

$$\mathrm{Ind}: \; K(\mathcal{M}) \to K(pt)$$

denote the index homomorphism in K-theory. For a positive integer k, we have the following formula which computes the Riemann-Roch number on $\mathcal{M}$ in terms of those on the lower rank moduli $\{P_j\}$

$$\mathrm{Ind}\, L_0^k = \mathrm{Res}_{z=0} \frac{1}{z} \sum_{P_j} \mathrm{Ind}_j \frac{i^*_{P_j} L^k}{\Lambda_z \nu_j}$$

where $z = e^{it}$, $\Lambda_z \nu_j$ is the equivariant Euler class of the normal bundle of P_j in K-theory and $\mathrm{Ind}_j : \; K(P_j) \to K(pt)$ is the index homomorphism on P_j in K-theory.

3. Sketch of proofs. 1). For the localization in equivariant cohomology. With a trick of [PW], the proof is basically the same as in §1. More precisely, for $u > 0$ large enough, by applying Bott localization method, we get

$$\int_{\mathcal{H}} e^{\tilde{\omega}} - \int_{\mathcal{M}} \frac{e^{\omega_0}}{e_t(\nu)} = e^{-tu} \int_{\mu^{-1}(u)/S^1} \frac{e^{\omega_u}}{t - F_u} + \sum_{P_j} \int_{P_j} \frac{i^*_{P_j} e^{\tilde{\omega}}}{e_t(\nu_j)}.$$

Here ω_u is the induced symplectic form on $\mu^{-1}(u)/S^1$ and F_u is the curvature of the circle bundle $\mu^{-1}(u) \to \mu^{-1}(u)/S^1$. The integral in the first term of the left hand side is a polynomial in u, since ω_u depends on u in polynomial and F_u is independent of u for u big enough.

Take t positive and let u go to infinity, the first term on the right hand side goes to zero. Note that $e_t(\nu) = \prod_{l=1}^{s}(t + x_l)$ where $\{x_l\}$ are the Chern rooots of the normal bundle ν of $\mathcal{M}$ in $\mathcal{H}$. Multiplying both sides by t^{s-1} and take the residues of both sides, we get the required formula.

For η some equivariant cohomology class of the universal bundles on $\mathcal{M}$, it is easy to extend the formula to the class like $e^{\tilde{\omega}}\eta$. See §3.

2). For equivariant K-theory, the proof is also the same as in §5 with a small modification. We consider the operator $\bar{\partial}^t_{L^k} = e^{t|\mu|^2}\bar{\partial}_{L^k}e^{-t|\mu|^2}$. When t is very large, its L^2 index is well-defined. As in §5, we consier the multiplicity formula as well as the fixed point formula for $\bar{\partial}^t_{L^k}$, we have the invariant part of the equivariant index of $\bar{\partial}^t_{L^k}$, denoted $[\mathrm{Ind}_{S^1}\bar{\partial}^t_{L^k}]^{S^1}$ on $\mathcal{H}$, is equal to the index of L_0 on $\mathcal{M}$, i.e.

$$[\mathrm{Ind}_{S^1}\bar{\partial}^t_{L^k}]^{S^1} = \mathrm{Ind}\, L_0^k.$$

Note that when restricted to $\mathcal{M}$, $\bar{\partial}^t_{L^k}$ is exactly $\bar{\partial}_{L_0^k}$ on $\mathcal{M}$, and

$$\mathrm{Ind}_{S^1}\bar{\partial}^t_{L^k} = \sum_{P_j}\mathrm{Ind}_j\frac{i^*_{P_j}L^k}{\Lambda_z\nu_j} + \int_{\mathcal{M}}\mathrm{Td}(T\mathcal{M})\frac{\mathrm{ch}\, L_0}{\prod_{l=1}^{s}(1 - z^{-1}e^{-x_l})}.$$

Then note that

$$[\mathrm{Ind}_{S^1}\bar{\partial}^t_{L^k}]^{S^1} = \mathrm{res}_{z=0}\frac{1}{z}\mathrm{Ind}_{S^1}\bar{\partial}^t_{L^k}$$

and that

$$\mathrm{res}_{z=0}\frac{1}{z}\int_{\mathcal{M}}\mathrm{Td}(T\mathcal{M})\frac{\mathrm{ch}\, L_0}{\prod_{l=1}^{s}(1 - z^{-1}e^{-x_l})} = 0$$

which gives the formula we want.

Remarks: 1. Note that, in the rank 2 case, the two K-theory localization formulas for vortex and Hitchin moduli spaces can be used to recover the proof of the Verlinde formula by Thaddeus [Th]. The formulas for equivariant cohomology can be used to give easy proofs of the Newstead conjectures on the vanishing of Chern and Pontryagin classes.

2. In any case, we see that the Guillemin-Sternberg multiplicity formula implies the Verlinde formula in a certain sense. One can also use symplectic cut [Le] to get the above formulas. The operator $\bar{\partial}^t_{L^k} = e^{t|\mu|^2}\bar{\partial}_{L^k}e^{-t|\mu|^2}$ was first considered in [TZ] in a different, but more general setting.

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Dept. of Math.
MIT
Cambridge, MA 02139.