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REMARKS ON THE GEOMETRY OF MODULI SPACES

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ABSTRACT. By using Yau's Schwarz lemma and the Quillen determinant line bundles, several results about fibered algebraic surfaces and the moduli spaces of curves are improved and reproved.

INTRODUCTION

Let $\pi : X \to B$ be a family of stable curves of genus $g > 1$, over a projective complex curve B. Let $\omega_{X/B}$ be the relative dualizing sheaf. Denote by $\mathfrak{M}_g$ the moduli space of smooth curves of genus g and by $\mathfrak{T}_g$ the corresponding Teichmüller space. The following results are important and well known in algebraic and complex geometry.

(1) $(\omega^2_{X/B})$ and $\det \pi_* \omega_{X/B}$ are always strictly positive for non-isotrivial families of stable curves.
(2) $(\omega^2_{X/B}) \leq (2g-2)(2q-2+s)$, where q is the genus of B and s is the number of singular fibers.
(3) If the family is non-isotrivial and B is CP^1, then $s \geq 3$; if B is an elliptic curve, then $s \geq 1$.
(4) The Weil-Petersson metric is Kähler and its Kähler class is rational.
(5) The compactified moduli space $\overline{\mathfrak{M}}_g$ is projective. Here we use the Deligne-Mumford compactification.
(6) The Teichmüller space $\mathfrak{T}_g$ is a domain of holomorphy.

In this paper, we give simple and elementary proofs of these results by using the differential geometry of $\mathfrak{M}_g$ and $\mathfrak{T}_g$ and make clear the geometric meanings of the quantities appearing above.

The first proof of (1) is by Arekelov [1], then by Ueno [16]. See also [17]. Result (2) was obtained by Vojta in [8] using the Bogomolov-Miyaoka-Yau inequality $c_1^2 \leq 3c_2$ for surfaces of general type. Our proof is much more elementary. By Parshin [18], we know that (2) already implies the Modell conjecture for a function field. Result (3) for CP^1 is due to Beauville [2]. Result (4) is due to Ahlfors and Wolpert. In fact we prove that (2) is a strict inequality for any non-isotrivial family of $g > 1$. Our methods are significantly different from their previous proofs. We use the Schwarz lemma and Quillen determinant line bundles which have been very well understood in differential geometry and string theory [10], [19].

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From our discussion, we can see that $(\omega_{X/B}^2)$ is in fact the degree of a natural positive line bundle on B. The Chern class of this line bundle is the Kähler class of the Weil-Petersson metric. We speculate that there should be a similar geometric interpretation of the positivity of $(\omega_{X/B}^2)$ in Arakelov theory for arithmetic surfaces [8]. We can also use our method to discuss families of Abelian varieties or Hodge structures [4].

It is interesting to note that the finiteness (or Mordell type) results for families are all intrinsically implied by the negative curvature properties of the corresponding moduli spaces. The height estimates are equivalent to the upper bound estimates of Green functions or Weil-Petersson metrics. We can also reduce Vojta's conjectural height estimate for a function field (see [8]) to an upper bound estimate of a Green function which agrees with the fact in number theory that the Green function is the ∞-adic height function. Therefore it should be interesting to see a number theory analogue of the Weil-Peterson metric.

We hope our method to prove (2), which is a simple application of the Schwarz lemma, can be used to derive $c_1^2 \leq 3c_2$ for surfaces of general type. So far we can get this for some special surfaces.

1. Some preparatory material

Let $\pi : X_g \to \mathfrak{T}_g$ be the universal curve over the Teichmüller space. The Poincaré metric on each fiber patches together to give a smooth metric on $\Omega_{X_g/\mathfrak{T}_g}$, the relative cotangent bundle. Then $\pi_*\Omega^{\otimes 2}_{X_g/\mathfrak{T}_g}$, the push-down of $\Omega^{\otimes 2}_{X_g/\mathfrak{T}_g}$, is the cotangent bundle of $\mathfrak{T}_g$. Recall that for any point $s \in \mathfrak{T}_g$

$$\pi_*\Omega^{\otimes 2}_{X_g/\mathfrak{T}_g}\Big|_s \triangleq H^0(X_s, \Omega^{\otimes 2}_{X_s})$$

where $X_s = \pi^{-1}(s)$.

There exists a natural inner product on $\pi_*\Omega^{\otimes 2}_{X_g/\mathfrak{T}_g}$ induced from the Poincaré metric on each fiber. This inner product induces the Weil-Petersson metric on $\pi_*\Omega^{\otimes 2}_{X_g/\mathfrak{T}_g}$ which is not complete. Its sectional curvature is bounded from above by $-\frac{1}{2\pi(g-1)}$ (see Wolpert [10]).

The Weil-Petersson metric is invariant under the action of modular group Γ_g. So it induces a metric on the moduli space of smooth curves of genus g, $\mathfrak{M}_g = \mathfrak{T}_g/\Gamma_g$. Here we consider $\mathfrak{M}_g$ as a V-manifold; see [7].

Let Δ_s be the Laplacian on $X_s = \pi^{-1}(s)$ acting on the functions of X_s and $\{\lambda_j(s)\}_{j=1}^{\infty}$ be its eigenvalues.

Let

$$\zeta(t) = \sum_{\lambda_j \neq 0} \lambda_j^{-t}(s),$$

which can be analytically extended to the whole complex t-plane, and define the Ray-Singer torsion

$$\det{}' \Delta_s = e^{-\zeta'(0)}.$$

Then on

$$\det \pi_*\omega_{X_g/\mathfrak{T}_g}\Big|_s \triangleq \Lambda^g H^0(X_s, \Omega_{X_s})$$

we have the Quillen metric

$$\|\cdot\|_Q^2 = \|\cdot\|_{L^2}^2 \cdot \det{}' \Delta_s$$

and the following Quillen formula [15]:

$$(*) \qquad c_1(\det \pi_* \omega_{X_g/\mathfrak{T}_g})_Q = \frac{\sqrt{-1}}{4\pi} \partial_s \bar{\partial}_s \log \frac{\det' \Delta_s}{\det \operatorname{Im} \tau} = \frac{1}{24\pi^2} \omega_{\mathrm{WP}}.$$

Here ω_{WP} is the Kähler form of the Weil-Petersson metric and $c_1(\cdot)_Q$ denotes the Chern form with respect to the Quillen metric. Also $\operatorname{Im}\tau$ is the imaginary part of the period matrix; in the other words, $(\det \operatorname{Im}\tau)$ is the L^2-norm on $\Lambda^g \pi_* \omega_{X_g/\mathfrak{T}_g}$ with respect to a basis of $H^0(S_x, \Omega_{X_s})$. For the most general Quillen formula for families of curves, see [3].

We also have the following equality of Wolpert [7]:

$$(**) \qquad \pi_* c_1^2(\omega_{X_g/\mathfrak{T}_g}) = \frac{1}{2\pi^2} \omega_{\mathrm{WP}}.$$

Here by π_*, we mean to take the integral along a generic fiber. The metric on $\omega_{X_g/\mathfrak{T}_g}$ is induced from the Poincaré metric of the fiber.

All of the above metrics are invariant under the action of the modular group Γ_g. So all of the above equalities still hold on the moduli space $\mathfrak{M}_g = \mathfrak{T}_g/\Gamma_g$ which is considered as a V-manifold. According to [12], as currents $(**)$ holds on $\overline{\mathfrak{M}}_g$. Let $\pi : \mathfrak{X}_g \to \mathfrak{M}_g$ be the universal curve. (We can define $\mathfrak{X}_g = X_g/\Gamma_g$ as a V-manifold.) In fact, $\overline{\mathfrak{M}}_g$, the Deligne-Mumford compactification of $\mathfrak{M}_g$, can also be obtained as the quotient of $\mathfrak{T}_g$ with partial boundary by Γ_g. Recall that, in the sense of Deligne-Mumford, $\overline{\mathfrak{M}}_g$ is obtained from $\mathfrak{M}_g$ by adding nodal curves. See [9], [12].

2. The proofs of (2) and (3)

Proof of (2). Let $X \xrightarrow{\pi} B$ be the family of stable curves. Then π induces a holomorphic map

$$B \xrightarrow{f} \overline{\mathfrak{M}}_g.$$

By definition

$$(\omega_{X/B}^2) = \int_B \pi_*(c_1^2(\omega_{X/B})) = \frac{1}{2\pi^2} \int_B f^* \bar{\omega}_{\mathrm{WP}}.$$

The second equality is the pull-back of $(**)$ by f. Here $\bar{\omega}_{\mathrm{WP}}$ is the extension of ω_{WP} to $\overline{\mathfrak{M}}_g$. Let Δ be those points over which the fibers are singular.

Assume the genus of B is $q > 1$, and if $q = 0$ or $q = 1$, then the number of points in Δ is bigger than or equal to 3 or 1, respectively. Then on $B - \Delta$ we have a natural complete metric, the Poincaré metric ω_{P} of constant curvature -1. The sectional curvature of the corresponding Weil-Petersson metric on $\mathfrak{M}_g$ is strictly bounded above by $-\frac{1}{2\pi(g-1)}$. So a simple application of the Schwarz lemma ([13], Theorem 2′) to the induced map

$$B - \Delta \xrightarrow{f} \mathfrak{M}_g$$

gives us

$$\int_B f^* \bar{\omega}_{\mathrm{WP}} = \int_{B-\Delta} f^* \omega_{\mathrm{WP}} < 2\pi(g-1) \int_{B-\Delta} \omega_{\mathrm{P}} = 2\pi(g-1) \cdot 2\pi(2q - 2 + s)$$

where q is the genus of B. The last step is Gauss-Bonnet. □

So the quantity $(2q-2+s)$ that appeared in the upper bound of $(\omega^2_{X/B})$ is in fact the negative Euler number of $B-\Delta$. We can compare this proof with the method in [8] and see that the upper bound of the sectional curvature of the Weil-Petersson metric and the Schwarz lemma are as effective as the Bogomolov-Miyaoka-Yau inequality for fibered surface. Note that we need to slightly generalize Yau's Schwarz lemma to V-manifolds, which can be easily done as in [7], Lemma 2. Actually since our domain manifold is one-dimensional and smooth, while the computations in [13] all occur on the domain manifold, both the methods and results in [13] can be easily carried over.

We note that $c_1(\omega_{X/B})$ was computed by Wolpert [11]. By [8] for any section $s: B\to X$, the height of this section is defined as

$$h(s)=\int_B s^*c_1(\omega_{X/B})=\int_{s(B)}c_1(\omega_{X/B}).$$

In [11], Wolpert obtained a formula for $c_1(\omega_{X/B})$ in terms of a Green function of the fiber. He also gave an upper bound estimate of this Green function which is independent of the genus of the fiber. This bound is not effective, but enough for a proof of the functional field Mordell conjecture. It might be possible to prove Vojta's conjectural inequality by giving a precise upper bound estimate of this Green function. The lower bound of $\int_B f^*\bar{\omega}_{\mathrm{WP}}$, which we are unable to get, will give the Xiao and Harris-Cornalba inequalities.

Proof of (3). This is very easy. In fact, given $X\xrightarrow{\pi}CP^1$, we have the induced map

$$CP^1-\Delta\xrightarrow{f}\mathfrak{M}_g$$

where Δ still denotes the singular set. Since CP^1 with one or two punctures have C as universal coverings, a simple application of the Schwarz lemma shows that f must be constant. One can prove the case of the elliptic curve with one puncture in the same way. □

3. The proofs of (1), (4), (5), and (6)

Proof of (1). Since $(\omega^2_{X/B})=\int_B\pi_*(c_1^2(\omega_{X/B}))=\frac{1}{2\pi^2}\int_B f^*\bar{\omega}_{\mathrm{WP}}$, the positivity of $(\omega^2_{X/B})$ is obvious. If $(\omega^2_{X/B})=0$, then $f^*\bar{\omega}_{\mathrm{WP}}=0$ and the image of the induced map $B\xrightarrow{f}\overline{\mathfrak{M}}_g$ must be one point. This means that $\pi: X\to B$ is isotrivial, i.e. trivial after a base change. Here since $\overline{\mathfrak{M}}_g$ is a V-manifold and ω_{WP} is considered as a metric on a V-manifold, we need a finite cover.

The positivity of $\det\pi_*\omega_{X/B}$ follows from the Noether formula [17]

$$\det\pi_*\omega_{X/B}=\frac{1}{12}((\omega^2_{X/B})+\delta)$$

where δ is the number of double points on bad fibers.

This means that $\det\pi_*\omega_{X/B}$ is more positive than $(\omega^2_{X/B})$. We can also use the Quillen formula $(*)$ to get this result. □

The statements in (1) were first proved by Arekelov using algebraic geometry [1]. This was an important step in his proof of the Shafarevich conjecture for the function field. Wolpert discussed the ampleness of $\omega_{X/B}$ in detail in [11] by using differential geometry.

Proofs of (4), (5) *and* (6). In fact, the equalities in $(*)$ already tell us a lot. First on $\mathfrak{T}_g$, the Teichmüller space, they tell us

$$\sqrt{-1}\partial\bar{\partial}\log\frac{\det'\Delta_2}{\det\operatorname{Im}\tau}=\frac{1}{6\pi}\omega_{\mathrm{WP}}.$$

This means that the globally defined function $\log\frac{\det'\Delta_s}{\det_{\operatorname{Im}\tau}}$ is the test function for the pseudoconvexity of $\mathfrak{T}_g$. This gives (6).

On $\mathfrak{M}_g$ we use

$$c_1(\det\pi_*\omega_{\mathfrak{X}_g/\mathfrak{M}_g})_Q=\frac{1}{24\pi^2}\omega_{\mathrm{WP}},$$

which tells us that $\det\pi*\omega_{\mathfrak{X}_g/\mathfrak{M}_g}$ is a positive line bundle on $\frac{1}{\pi^2}\mathfrak{M}_g$ on one hand and ω_{WP} is Kähler and rational on the other hand, which implies (4).

For (5), as shown in [9], the key point is to find a positive line bundle on $\overline{\mathfrak{M}}_g$. We content ourselves in just giving such a line bundle. First note that on $\overline{\mathfrak{M}}_g$ we have the equality of currents

$$12c_1(\det\pi_*\omega_{\overline{\mathfrak{X}}_g/\overline{\mathfrak{M}}_g})-\delta=\frac{1}{2\pi^2}\bar{\omega}_{\mathrm{WP}}$$

where δ is the current corresponding to the first Chern class of the line bundle given by the compactification divisor $\Delta=[\overline{\mathfrak{M}}_g-\mathfrak{M}_g]$.

There are at least three ways to prove this equality. The first is to use the formula in [3] which gives

$$\pi_*c_1^2(\omega_{\overline{\mathfrak{X}}_g/\mathfrak{M}_g})=12c_1(\det\pi_*\omega_{\overline{\mathfrak{X}}_g/\overline{\mathfrak{M}}_g})_Q-\delta$$

together with $(**)$ which also holds on $\overline{\mathfrak{M}}_g$ by Wolpert [10], [11]. That is

$$\pi_*c_1^2(\omega_{\overline{\mathfrak{X}}_g/\overline{\mathfrak{M}}_g})=\frac{1}{2\pi^2}\bar{\omega}_{\mathrm{WP}}.$$

The second is to use a result of Wolpert in [12]. The third is to directly compute the degeneracy of $\det'\Delta_s$ which is well known in string theory [19]. So $\bar{\omega}_{\mathrm{WP}}$ is still rational on $\overline{\mathfrak{M}}_g$. This also proves the famous ampleness of $12\lambda-\delta$. Here we still use δ to denote the line bundle corresponding to Δ, and $\lambda=\det\pi_*\omega_{\overline{\mathfrak{X}}_g/\overline{\mathfrak{M}}_g}$ is the so-called Hodge line bundle. So $\bar{\omega}_{\mathrm{WP}}$ or $12\lambda-\delta$, both positive and rational, can be used to get the required embedding of $\mathfrak{M}_g$ and $\overline{\mathfrak{M}}_g$ into CP^N. For V-manifolds, the embedding property was discussed by Baily. A quite different way to get a positive line bundle on the moduli space, also using Weil-Petersson metric, was given by Wolpert in a series of papers. □

As one can see, many interesting properties of the moduli space are direct consequences of the Quillen formula. Actually it is possible to use the Quillen formula to prove the projectivities of other moduli spaces, since we have similar formula for higher dimensional manifolds. We can also get some height estimates. For example, for $A_g\to\mathfrak{A}_g$, the universal family of Abelian varieties over the corresponding moduli space, we have the formula

$$c_1(\omega_{A_g/\mathfrak{A}_g})=\omega_{\mathrm{WP}}$$

where $(\omega_{A_g/\mathfrak{A}_g}) = \Lambda^g \Omega^1_{A_g/\mathfrak{A}_g}$ and ω_{WP} is the Kähler class of Weil-Petersson metric on $\mathfrak{A}_g$. In fact, we have a more general formula for the moduli space of compact complex manifolds with $c_1 = 0$ (see [6]). Then by our method, the degree estimate in [4] is equivalent to the estimate of the degree of ω_{WP}. It is easy to get effective height estimates by using the Schwarz lemma and the curvature computation of Griffiths-Schmid. We leave these as exercises to the readers.

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References

1. S. Arakelov, *Families of algebraic curves with fixed degeneracies*, Izv. Akad. Nauk SSSR Ser. Math. **35** (1971), no. 6, 1269–1293. MR **48:**298
2. A. Beauville, *Le nombre minimun de fibres singulieres d'une courbe stable sur P^1*, in Séminaire sur les pinceaux de courbes degenre au moins deux, ed. L. Szpiro, Asterisque **86** 97–108.
3. J. M. Bismut and J. B. Bost, *Fibre determinant, metriques de Quillen et degenerescence des courbes*, Acta Math. **165** (1990), 1–103. MR **91h:**58122
4. G. Faltings, *Arakelov theorem for abelian varieties*, Invent. Math. **73** (1983), 337–348. MR **85m:**14061
5. M. Gromov, *Kahler hyperbolicity and L_2-Hodge theory*, J. Differential Geom. **33** (1991), 263–292. MR **92a:**58133
6. G. Tian, *Smoothness of the universal deformation space of compace Calabi-Yau manifolds and its Weil-Peterson metric*, Mathematical Aspects of String Theory (S. T. Yau, ed.), World Scientific, Singapore, 1987, pp. 629–646. CMP 20:04
7. G. Tian and S. T. Yau, *Existence of Kahler-Einstein metrics on complete manifolds and their applications to algebraic geometry*, Mathematical Aspects of String Theory (S. T. Yau, ed.), World Scientific, Singapore, 1987, pp. 574–628. CMP 20:04
8. P. Vojta, *Diophantine inequalities and Arakelov theory*, in S. Lang, Introduction to Arakelov Theory, Springer-Verlag, 1988, pp. 155–178. MR **89m:**11059
9. S. Wolpert, *On obtaining a positive line bundle from the Weil-Peterson class*, Amer. J. Math. **107** (1985), 1485–1507. MR **87f:**32058
10. ———, *Chern forms and the Riemann tensor for the moduli space of curves*, Invent. Math. **85** (1985), 119–145. MR **87j:**32070
11. ———, *The hyperbolic metric and the geometry of the universal curve*, J. Differential Geom. **31** (1990), 417–427. MR **91a:**32030
12. ———, *Homology theory of moduli space of stable curves*, Ann. of Math. (2) **118** (1983), 491–523. MR **86h:**32036
13. S. T. Yau, *A general Schwarz lemma for Kahler manifolds*, Amer. J. Math. **100** (1978), 197–203. MR **58:**6370
14. P. G. Zograf, *The Liouville action on moduli spaces and uniformization on degenerating Riemann surfaces*, Leningrad Math. J. **1** (1990), 941–965. MR **91c:**32015
15. P. Zograf and T. Takhtajan, *A potential of the Weil-Peterson metric on the Torelli space*, Zap. Nauchn. Sem. Leningrad Otdel. Mat. Inst. Steklov. (LOMI) **160** (1987). MR **89a:**32022
16. K. Ueno, *Kodaira dimension of certain fiber spaces*, Complex Analysis and Algebraic Geometry, Iwanami-Shoten, Tokyo, 1977, pp. 279–292. MR **56:**5556

17. W. Barth, C. Peters, and A. Ven de Ven, *Compact complex surfaces*, Springer-Verlag, Berlin and New York, 1984. MR **86c:**32026
18. A. N. Parshin, *Algebraic curves over functional fields* I, Math. USSR Izv. **2** (1968), 1145–1170. MR **41:**1740
19. E. D'Hoker and D. H. Phong, *Geometry of quantum strings*, Mathematical Aspects of String Theory (S. T. Yau, ed.), World Scientific, Singapore, 1987, pp. 29–59. CMP 20:04

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