Mathematical Results

Inspired by Physics

Results of three different types in geometry and topology:

(1) **Elliptic Genera**: Vanishing and Rigidity Theorems.

(2) **Moduli Spaces**: Formulas for Intersection Numbers.

(3) **Mirror Principle**: Formulas for Counting Curves in CY Manifolds.

Common Features:

A. Physicists made conjectures based on physical principles, or formal mathematical arguments.

Localizations on infinite dimensional spaces: Path integrals.

B. The mathematical proofs depend on **localization techniques**, combining with various parts of mathematics:

Modular forms, Heat kernels, Symplectic geometry, Various moduli spaces.

C. New methods of proofs inspired more new results.

(1). Elliptic Genus:

Only discuss a vanishing theorem of the Witten genus, which is the index of the Dirac operator on loop space. This is a loop space analogue of the Atiyah-Hirzebruch vanishing theorem.

Will also discuss a loop space analogue of a vanishing theorem of Lawson-Yau.

There are much more general new results to be mentioned.

Let M be a compact smooth spin manifold with an S^1-action, D be the Dirac operator.

Theorem: (AH 70) $\operatorname{Ind} D = \widehat{A}(M) = 0$.

Remark: $K3$ surface does not allow nontrivial smooth S^1-action.

Loop space: $LM = \text{Maps}\{S^1 \to M\}$.

Dirac operator on LM: (S^1-equivariant)

$$D^L = D \otimes \bigotimes_{n=1}^{\infty} S_{q^n} TM = \sum_{n=0} D \otimes V_n \; q^n$$

where V_n combinations of $S^j(TM)$ from symmetric operation:

$$S_t E = 1 + tE + t^2 S^2 E + \cdots .$$

D^L=Infinitely many operators: $D,\ D \otimes TM, \cdots$.

Theorem: (L 92) If $p_1(M)_{S^1} = n\,\pi^* u^2$, then the Witten genus vanishes: $\text{Ind}\, D^L = 0$.

Here $p_1(M)_{S^1}$ is the equivariant first Pontrjagin class and u the generator of $H^*(BS^1, \mathbb{Z})$.

$\pi : \quad M \times_{S^1} ES^1 \to BS^1$ natural projection.

The condition on $p_1(M)_{S^1}$ is equivalent to that the S^1 preserves the spin structure of LM.

If S^3 acts on M, then $p_1(M)_{S^1} = n\,\pi^* u^2$ is equivalent to $p_1(M) = 0$, ($\simeq LM$ is spin).

Corollary: Assume S^3 acts on M and $p_1(M) = 0$, then the Witten genus, $\operatorname{Ind} D^L$, vanishes.

Compare with Lawson-Yau's theorem:

Theorem: (LY 73) Assume S^3 acts on M, then $\operatorname{Ind} D = 0$.

The proof combines the Atiyah-Bott-Segal-Singer fixed point formula with Jacobi forms.

Liu 92: More new rigidity, vanishing and divisibility results involving general loop group representations can be proved in a similar way:

Kac-Weyl character formula has modular property.

Level 1 representations of loop groups $\Rightarrow$ Witten conjectures (1986) on rigidity of elliptic genera: Bott-Taubes, Hirzebruch, Krichever, Landweber-Stong, Ochanine. ($\simeq$ Signature on loop space.)

Recent works:

With Ma and Zhang: Family; Foliation;

With Dong and Ma: Orbifolds; Vertex operator algebra bundles.

93: Miraculous cancellation formula for $8k+4$ dimensional manifolds: At top degree

$$L(TM) = 2^3 \sum_{j=0}^{k} 2^{6k-6j} \widehat{A}(TM) \,\mathrm{ch}\, B_j$$

where B_j from operations of TM. This is an identity on differential forms $\Rightarrow$ divisibility, eta-invariants, holonomy of determinant line bundles.

More general formulas involving vector bundles can be proved from **modular invariance**.

Alveraz-Gaume and Witten(1982): $k = 1$. Important in string theory.

(2). Moduli Spaces

Moduli spaces of flat connections on Riemann surfaces: From 50's to 90's, Indian school; GIT theory; Atiyah-Bott, Witten: Gauge theory; Donaldson and English school....

The most effective way to compute intersection numbers from heat kernel (L 95-96): Prove formulas of Witten derived from path integrals.

$\mathcal{M}_u$: Moduli space of flat G-connections on principal G bundle on a Riemann surface S with boundary. $u \in Z(G)$, the center.

$\mathcal{M}_c$: Moduli space of flat connections on a G bundle on S with holonomy around the boundary $= c \in G$ (close to u).

Assume moduli smooth, G simply connected. Method works for general cases.

Refined Witten Fromula:

Theorem:(L 96) We have equality:

$\int_{\mathcal{M}_u} p(\sqrt{-1}\Omega)\, e^{\omega_u} = |Z(G)| \frac{|G|^{2g-2}}{(2\pi)^{2N_u}}.$

$\lim_{c \to u} \lim_{t \to 0} \sum_{\lambda \in P_+} \frac{\chi_\lambda(c)}{d_\lambda^{2g-1}}\, p(\lambda + \rho)\, e^{-t p_c(\lambda)}.$

Notations:

$p(\sqrt{-1}\Omega)$: Pontrjagin class of $T\mathcal{M}_u$ associated to a Weyl-invariant polynomial p.

P_+: all irreducible representations of G, as a lattice in t^*.

χ_λ: character of λ, d_λ its dimension.

$p_c(\lambda) = |\lambda + \rho|^2 - |\rho|^2$. $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$.

Δ^+: positive roots.

N_u: complex dimension of $\mathcal{M}_u$; g: genus of S.

ω_u: canonical symplectic form on $\mathcal{M}_u$ induced from Poincare duality of cohomology.

Remark:

Derivative with respect to c + Heat kernel $\Rightarrow$ Symplectic volume:

$\mathbf{Vol}(\mathcal{M}_c) = \int_{\mathcal{M}_c} e^{\omega_c}$,

where ω_c canonical symplectic form on $\mathcal{M}_c$, is a (piecewise) polynomial in C of degree at most $(2g-1)\,|\Delta^+|$ ($c = u \exp C$)

$\Rightarrow$ Vanishing results of intersection numbers.

Proof uses the holonomy model of the moduli space and heat kernel on G:

Consider map: $f: \quad G^{2g} \times O_c \to G$ with

$$f(x_1, \cdots, y_g; z) = \prod_{j=1}^{g} [x_j, y_j] z.$$

O_c: conjugacy class through (generic) $c \in G$.

Moduli space $\mathcal{M}_c = f^{-1}(e)/G$.

Heat kernel on G:

$H(t, x, y) = \frac{1}{|G|} \sum_{\lambda \in P_+} d_\lambda \cdot \chi_\lambda(xy^{-1}) e^{-tp_c(\lambda)}$

where $|G|$: volume of G.

Compute the integral

$$I(t) = \int_{h \in G^{2g} \times O_c} H(t, c, f(h)) dh$$

in two ways: Local and Global,

Local: As $t \to 0$, $I(t)$ localizes to integral on $\mathcal{M}_c$, which is the symplectic volume (= Reidemeister torsion = the RS-torsion). Poincare duality for both symplectic form and torsion(Witten, B-L, Milnor, Johnson).

Global: Orthogonal relations for the characters gives the infinite sum.

Take derivative with respect to c + Symplectic geometry relating ω_c to $\omega_u \Rightarrow$ the final formula.

Remarks:

(1). Similar results for more boudary components moduli.

S: Riemann surface with s boundary components.

$\mathcal{M}_{\mathbf{c}}$: Moduli of flat connections with holonomy $c_1, \cdots, c_s \in G$ around the corresponding boundaries.

$\omega_{\mathbf{c}}$: canonical symplectic form from Poincare duality.

More general refined Witten formula:

Theorem(L 96): We have equality:

$$\int_{\mathcal{M}_{\mathbf{c}}} p(\sqrt{-1}\Omega) e^{\omega_{\mathbf{c}}} = |Z(G)| \frac{|G|^{2g-2+s} \prod_{j=1}^{s} j(c_j)}{(2\pi)^{2N_{\mathbf{c}}} \prod_{j=1}^{s} |Z_{c_j}|} \cdot$$

$$\lim_{t \to 0} \sum_{\lambda \in P_+} \frac{\prod_{j=1}^{s} \chi_\lambda(c_j)}{d_\lambda^{2g-2+s}} p(\lambda + \rho) e^{-t p_c(\lambda)}.$$

Here $N_{\mathbf{c}}$: the complex dimension of $\mathcal{M}_{\mathbf{c}}$;

$p(\sqrt{-1}\Omega)$: Pontrjagin class of $\mathcal{M}_{\mathbf{c}}$;

Z_{c_j}: centralizer of c_j;

$j(c_j)$: the Weyl denominator.

Taking derivatives with respect to the c_j's: get intersection numbers involving the other generators of the cohomology ring of $\mathcal{M}_{\mathbf{c}}$.

(2). The integrals we computed contain all the information for Verlinde formula. Bismut-Labourie: Rewrite infinite sum as "finite sum": residues.

Derivatives of Volume + Residues $\Rightarrow$ Verlinde.

(3). Such method applies to more general situation like moment maps, which gives the nonabelian localization formula; fundamental groups of 3-manifolds....

(3) Mirror Principle.

X: Projective manifold.

$\mathcal{M}_{g,k}(d, X)$: moduli space of stable maps of genus g and degree d with k marked points into X, modulo the obvious equivalence.

Points in $\mathcal{M}_{g,k}(d, X)$ are triples: $(f; C; x_1, \cdots, x_k)$:

$f : C \to X$: degree d holomorphic map;

$x_1, \cdots, x_k$: k distinct smooth points on the genus g curve C.

$f_*([C]) = d \in H_2(X, , \mathbf{Z})$: identified as integral index $(d_1, \cdots, d_n)$ by choosing a basis of $H_2(X, , \mathbf{Z})$ (dual to the Kahler classes).

Virtual fundamental cycle of Li-Tian, (Behrend-Fantechi): $LT_{g,k}(d, X)$, a homology class of the expected dimension in $\mathcal{M}_{g,k}(d, X)$.

Consider the case $k = 0$ first.

V: concavex bundle on X, direct sum of a positive and a negative bundle on X.

V induces sequence of vector bundles V_d^g on $\mathcal{M}_{g,k}(d, X)$: $H^0(C, f^*V) \oplus H^1(C, f^*V)$.

b: a multiplicative characteristic class.

Problem: Compute $K_d^g = \int_{LT_{g,0}(d,X)} b(V_d^g)$.

Mirror Principle: Compute

$$F(T, \lambda) = \sum_{d,\, g} K_d^g \, \lambda^g \; e^{d \cdot T}$$

in terms of hypergeometric series.

Here λ, $T = (T_1, \cdots, , T_n)$ formal variables.

Balloon manifold X: projective manifold with complex torus action and isolated fixed points.

$H = (H_1, \cdots, H_n)$: a basis of equivariant Kahler classes.

X is called a balloon manifold (GKM) if:

(1). $H(p) \neq H(q)$ for any two fixed points p, $q \in X$;

(2). The tangent bundle T_pX has linearly independent weights for any fixed point $p \in X$.

Complex 1-dimensional orbits: balloons $\simeq$ copies of $\mathbf{P}^1$.

V: fixed splitting type when restricted to each balloon.

Theorem: (LLY 97) Mirror principle holds for balloon manifolds and concavex bundles.

Mirror formulas: $b =$ Euler class, the genus $g = 0$.

Mirror principle implies that mirror formulas actually hold for very general manifolds such as Calabi-Yau complete intersections in toric manifolds and in compact homogeneous manifolds

$\Rightarrow$ All the mirror formulas for counting rational curves predicted by string theorists.

Mirror principle holds even for non-Calabi-Yau and for certain local complete intersections.

Mirror principle for counting higher genus curves: need to find the explicit hypergeometric series.

Example: Consider toric manifold X and $g = 0$. $D_1, .., D_N$: toric invariant divisors.

$V = \bigoplus_j L_j$, with $c_1(L_j) \geq 0$ and $c_1(X) = c_1(V)$.

$\langle \cdot, \cdot \rangle$: pairing of homology and cohomology classes.

b: the Euler class e; $\Phi(T) = \sum_d K_d^0 \, e^{d \cdot T}$.

Hypergeometric series

$$HG[B](t) = e^{-H \cdot t} \sum_d \prod_j \prod_{k=0}^{\langle c_1(L_j), d \rangle} (c_1(L_j) - k) \cdot$$

$$\frac{\prod_{\langle D_a, d \rangle < 0} \prod_{k=0}^{-\langle D_a, d \rangle - 1} (D_a + k)}{\prod_{\langle D_a, d \rangle \geq 0} \prod_{k=1}^{\langle D_a, d \rangle} (D_a - k)} \; e^{d \cdot t}.$$

Corollary: There are explicitly computable functions $f(t)$, $g(t)$, such that

$$\int_X \left(e^f HG[B](t) - e^{-H \cdot T} e(V) \right) = 2\Phi - \sum_j T_j \frac{\partial \Phi}{\partial T_j}$$

where $T = t + g(t)$.

Φ: determined uniquely from equation.

f, g: from expansion of $HG[B](t)$.

V: can be more general concavex bundles with splitting type.

Example: Calabi-Yau quintic.

$V = \mathcal{O}(5)$ on $X = \mathbf{P}^4$ and the hypergeometric series is:

$$HG[B](t) = e^{H\,t} \sum_{d=0}^{\infty} \frac{\prod_{m=0}^{5d}(5H+m)}{\prod_{m=1}^{d}(H+m)^5}\, e^{d\,t},$$

H: hyperplane class on $\mathbf{P}^4$; t: parameter.

Introduce

$\mathcal{F}(T) = \frac{5}{6}T^3 + \sum_{d>0} K_d^0 \, e^{dT}$.

Algorithm: Take expansion in H:

$HG[B](t) = H\{f_0(t)+f_1(t)H+f_2(t)H^2+f_3(t)H^3\}$.

Candelas Formula: With $T = f_1/f_0$,

$$\mathcal{F}(T) = \frac{5}{2}(\frac{f_1}{f_0}\frac{f_2}{f_0} - \frac{f_3}{f_0}).$$

Local mirror symmetry: V concave bundle: $\simeq$ Geometric engineering in string theory. (Canonical bundle on del Pezzo surfaces, open CY).

Hypergeometric series = periods of elliptic curves: Seiberg-Witten curves.

Trivial example: $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ on $\mathbf{P}^1$: multiple cover formula.

Key ingredients for the proof:

(1). Euler data;

(2). Linear and non-linear sigma model;

(3). Balloon and hypergeometric Euler data.

Nonlinear sigma model: stable map moduli;

Linear sigma model: simple moduli space.

Apply **functorial localization formula** to the equivariant collapsing map between the two sigma models, and to the evaluation maps.

Push computations to the simple spaces.

Hypergeometric series naturally appear from localizations on the linear sigma models and at the smooth fixed points in the moduli spaces.

Generalized Euler data includes general Gromov-Witten invariants: compute integrals of the form:

$$K^g_{d,k} = \int_{LT_{g,k}(d,X)} \prod_j ev_j^* \omega_j \cdot b(V^g_d)$$

where $\omega_j \in H^*(X)$.

Generalized Mirror Principle: compute such series in terms of hypergeometric series. Being developed together with mirror principle for counting discs, higher genus curves....

Remarks:

(1). Most important contribution: Candelas and his collaborators, Witten, Vafa, Warner, Greene, Morrison, Plesser, and many others: physical theory of mirror symmetry, computations used mirror manifolds and their periods.

(2). Related mathematical works: Yau and his collaborators Lian, Hosono, Klemm; Katz, Kontsevich, Givental, Gathman, Bertram....

(3). More conjectures from physics can be approached by localization techniques.