

Mathematical Results Inspired by Physics

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Abstract

I will discuss results of three different types in geometry and topology. (1) General vanishing and rigidity theorems of elliptic genera proved by using modular forms, Kac-Moody algebras and vertex operator algebras. (2) The computations of intersection numbers of the moduli spaces of flat connections on a Riemann surface by using heat kernels. (3) The mirror principle about counting curves in Calabi-Yau and general projective manifolds by using hypergeometric series.

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1. Introduction

The results I will discuss are all motivated by the conjectures of physicists, without which it is hard to imagine that these results would have appeared. In all these cases the new methods discovered during the process to prove those conjectures often give us many more surprising new results. The common feature of the proofs is that they all depend on localization techniques built upon various parts of mathematics: modular forms, heat kernels, symplectic geometry, and various moduli spaces.

Elliptic genera were invented through the joint efforts of physicists and mathematicians [17]. Actually in Section 2 I will only discuss in detail a vanishing theorem of the Witten genus, which is the index of the Dirac operator on loop space. This is a loop space analogue of the famous Atiyah-Hirzebruch vanishing theorem. It was discovered in the process of understanding the Witten rigidity conjectures for elliptic genera. A loop space analogue of a famous theorem of Lawson-Yau for non-abelian Lie group actions will also be discussed.

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Moduli spaces of flat connections on Riemann surfaces have been studied for many years in various subjects of mathematics [2]. The computations of the intersection numbers on such moduli spaces have been among the central problems in the subject. In Section 3 I will discuss a very effective way to compute the most interesting intersection numbers by using the localization property of heat kernels. This proves several beautiful formulas conjectured by Witten [38]. We remark that these intersection numbers include those needed for the Verlinde formula.

In Section 4 I discuss some remarkable formulas about counting curves in projective manifolds, in particular in Calabi-Yau manifolds. I will discuss the mirror principle, a general method developed in [27]-[30] to compute characteristic classes and characteristic numbers on moduli spaces of stable maps in terms of hypergeometric series. The mirror formulas from mirror symmetry correspond to the computations of the Euler numbers. Mirror principle computes quite general Hirzebruch multiplicative classes such as the total Chern classes.

2. Elliptic Genera

Let M be a compact smooth spin manifold with a non-trivial S^1 -action, D be the Dirac operator on M . Atiyah and Hirzebruch proved that in such a situation the index of the Dirac operator $\text{Ind } D = \hat{A}(M) = 0$, where $\hat{A}(M)$ is the Hirzebruch \hat{A} -genus [3]. One interesting application of this result is that a $K3$ surface does not allow any non-trivial smooth S^1 -action, because it has non-vanishing \hat{A} -genus.

Let LM be the loop space of M . LM consists of smooth maps from S^1 to M . There is a natural S^1 -action on LM induced by the rotation of the loops, whose fixed points are the constant loops which is M itself. Witten formally applied the Atiyah-Bott-Segal-Singer fixed point formula to the Dirac operator on LM , from which he derived the following formal elliptic operator [36]:

$$D^L = D \otimes \bigotimes_{n=1}^{\infty} S_{q^n} TM = \sum_{n=0}^{\infty} D \otimes V_n q^n$$

where q is a formal variable and for a vector bundle E ,

$$S_q E = 1 + q E + q^2 S^2 E + \cdots$$

is the symmetric operation and V_n is the combinations of the symmetric products $S^j(TM)$'s by formal power series expansion. So D^L , which is called the Dirac operator on loop space, actually consists of an infinite series of twisted Dirac operators with the pure Dirac operator D as the degree 0 term. The index of D^L , denoted by $\text{Ind } D^L$, is called the Witten genus. The loop space analogue of our vanishing theorem is the following:

Theorem 2.1: ([21]) *Let M be a spin manifold with non-trivial S^1 -action. Assume $p_1(M)_{S^1} = n\pi^*u^2$ for some integer n , then the Witten genus vanishes: $\text{Ind } D^L = 0$.*

Here $p_1(M)_{S^1}$ is the equivariant first Pontrjagin class and u is the generator of the cohomology group of the classifying space BS^1 , and $\pi : M \times_{S^1} ES^1 \rightarrow BS^1$ is the natural projection from the Borel construction.

This theorem implies that under the extra condition on the first Pontrjagin class, we have infinite number of elliptic operators with vanishing indices. The condition on the first equivariant Pontrjagin class is equivalent to that the S^1 -action preserves the spin structure of LM . If we have a non-abelian Lie group acts on M non-trivially, then for an S^1 subgroup, the condition $p_1(M)_{S^1} = n\pi^*u^2$ is equivalent to $p_1(M) = 0$ which implies that LM is spin. As an easy consequence, we get:

Corollary 2.2: *Assume a non-abelian Lie group acts on the spin manifold M non-trivially and $p_1(M) = 0$, then the Witten genus, $\text{Ind } D^L$, vanishes.*

This corollary should be considered as a loop space analogue of a result of Lawson-Yau in [18], which states that if a non-abelian Lie group acts on the spin manifold M non-trivially, then $\text{Ind } D = 0$. Our results motivated Hoehn and Stolz to conjecture that, for a compact spin manifold M with positive Ricci curvature and $p_1(M) = 0$, the Witten genus vanishes. So far all of the known examples have non-abelian Lie group action, therefore our results applies. It should be interesting to see how to combine curvature with modular forms to get vanishing results.

The proof of Theorem 2.1 is an interesting combination of the Atiyah-Bott-Segal-Singer fixed point formula with Jacobi forms. The magic combination of geometry and modular invariance implies the vanishing of the equivariant index of D^L . Similar idea can be used to prove many more rigidity, vanishing and divisibility results for D^L twisted by bundles constructed from loop group representations. Such operators can be viewed as twisted Dirac operators on loop space. See [20] and [21]. In these cases the Kac-Weyl character formulas came into play. If we take the level 1 representations of the loop group of the spin group in our general rigidity theorem, we get the Witten conjectures on the rigidity of elliptic genera [36], which were proved by Taubes [35], Bott-Taubes [8], Hirzebruch [15], Krichever, Landweber-Stong, Ochanine for various cases.

Our method can actually go very far. Recently in [33] we proved rigidity and vanishing theorems for families of elliptic genera and the Witten genus. In [32] we proved similar theorems for foliated manifolds. In [10] such theorems were generalized to orbifolds. More recently in [11] we have proved a far general rigidity theorem for D^L twisted by vertex operator algebra bundles.

If we apply the modular invariance argument to the non-equivariant elliptic genera, we get a general formula which expresses the Hirzebruch L -form in terms of the twisted \hat{A} -forms [22]. A 12 dimensional version of this formula, due to Alveraz-Gaume and Witten, called the miraculous cancellation formula, had played important role in the development of string theory. This formula has many interesting mathematical consequences involving the eta-invariants. We refer the reader to [22] and [23].

3. Moduli Spaces

Let G be a compact semi-simple Lie group and \mathcal{M}_u be the moduli space of flat connections on a principal flat G -bundle P on a Riemann surface S with boundary, where $u \in Z(G)$ is an element in the center. Here for simplicity we first discuss the

case when S has one boundary component, G is simply connected and the moduli space is smooth. A point in \mathcal{M}_u is an equivalence class of flat connection on P with holonomy u around the boundary. In general we let \mathcal{M}_c denote the moduli space of flat connections on P with holonomy around the boundary to be $c \in G$ which is close to u , or equivalently in the conjugacy class of c . The following formula is essentially a refined version of the formula [38] which Witten derived from the path integrals on the space of connections.

Theorem 3.1: ([24], [25]), *We have the following identity:*

$$\int_{\mathcal{M}_u} p(\sqrt{-1}\Omega)e^{\omega_u} = |Z(G)| \frac{|G|^{2g-2}}{(2\pi)^{2N_u}} \cdot \lim_{c \rightarrow u} \lim_{t \rightarrow 0} \sum_{\lambda \in P_+} \frac{\chi_\lambda(c)}{d_\lambda^{2g-1}} p(\lambda + \rho) e^{-tp_c(\lambda)}.$$

The notations in the above formula are as follows: ω_u is the canonical symplectic form on \mathcal{M}_u induced by Poincare duality on S ; $p(\sqrt{-1}\Omega)$ is a Pontrjagin class of the tangent bundle $T\mathcal{M}_u$ of the moduli space associated to the symmetric polynomial p ; P_+ is the set of irreducible representations of G identified as a lattice in \mathcal{T}^* which is the dual Lie algebra of the maximal torus T of G ; $p_c(\lambda) = |\lambda + \rho|^2 - |\rho|^2$ where $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$ with respect to the Killing form, and Δ^+ denotes the set of positive roots; χ_λ and d_λ are respectively the character and dimension of λ ; $|G|$ denotes the volume of G with respect to the bi-invariant metric induced from the Killing form; $|Z(G)|$ denotes the number of elements in the center $Z(G)$ of G and finally N_u is the complex dimension of \mathcal{M}_u .

The starting point for the proof of this theorem is to use the holonomy model of the moduli space and the explicit heat kernel on G . We consider the holonomy map $f : G^{2g} \times O_c \rightarrow G$ with $f(x_1, \dots, y_g; z) = \prod_{j=1}^g [x_j, y_j]z$ where O_c is the conjugacy class through the generic point $c \in G$. It is well-known that the moduli space is given by $\mathcal{M}_c = f^{-1}(e)/G$ where G acts on $G^{2g} \times O_c$ by conjugation.

We have the explicit expression for the heat kernel on G :

$$H(t, x, y) = \frac{1}{|G|} \sum_{\lambda \in P_+} d_\lambda \cdot \chi_\lambda(xy^{-1}) e^{-tp_c(\lambda)}$$

where $x, y \in G$ are two points. The key idea is to consider the integral

$$I(t) = \int_{h \in G^{2g} \times O_c} H(t, c, f(h)) dh$$

where dh denotes the induced bi-invariant volume form on $G^{2g} \times O_c$. We compute $I(t)$ in two different ways. First as $t \rightarrow 0$, $I(t)$ localizes to an integral on \mathcal{M}_c , which is the symplectic volume of \mathcal{M}_c with respect to the canonical symplectic form induced by the Poincare duality on the cohomology groups of S with values in the adjoint Lie algebra bundle. To prove this we used the beautiful observation of Witten [37] that the symplectic volume form of \mathcal{M}_c is the same as the Reidemeister torsion which arises from the Gaussian integral in the heat kernel.

On the other hand the orthogonal relations among the characters of the representations of G easily give us the infinite sum. In summary we have obtained

the following more precise version of Witten's beautiful formula for the symplectic volume of the moduli space,

Proposition 3.2: ([24]) *As $t \rightarrow 0$, we have*

$$\int_{\mathcal{M}_c} e^{\omega_c} = |Z(G)| \frac{|G|^{2g-1} |j(c)|}{(2\pi)^{2N_c} |Z_c|} \sum_{\lambda \in P_+} \frac{\chi_\lambda(c)}{d_\lambda^{2g-1}} e^{-tp_c(\lambda)} + O(e^{-\delta^2/4t}).$$

Here δ is any small positive number, $|Z_c|$ is the volume of the centralizer Z_c of c , $j(c) = \prod_{\alpha \in \Delta^+} (e^{\sqrt{-1}\alpha(C)/2} - e^{-\sqrt{-1}\alpha(C)/2})$ is the Weyl denominator, and N_c is the complex dimension of \mathcal{M}_c .

To get the intersection numbers from the volume formula, we take derivatives with respect to C where $c = u \exp C$. This is another key observation. By using the relation between the symplectic form on \mathcal{M}_c and that on \mathcal{M}_u , and then taking the limits we arrive at the formula in Theorem 3.1. For the details see [24] and [25]. Another easy consequence of the method is that the symplectic volume of \mathcal{M}_c is a piecewise polynomial of degree at most $2g|\Delta^+|$ in $C \in \mathcal{T}$ from which we get certain very general vanishing theorems for those integrals when the degree of the polynomial p is relatively large [25].

Similar results for moduli spaces when S has more boundary components can be obtained in the same way [25]. More precisely, assume S has s boundary components and consider the moduli space of flat connections on the principal G bundle P with holonomy $c_1, \dots, c_s \in G$ around the corresponding boundaries. Let \mathcal{M}_c denote the moduli space and ω_c denote the canonical symplectic form. Then we have

Theorem 3.3 ([25]) *The following formula holds:*

$$\int_{\mathcal{M}_c} p(\sqrt{-1}\Omega) e^{\omega_c} = |Z(G)| \frac{|G|^{2g-2+s} \prod_{j=1}^s j(c_j)}{(2\pi)^{2N_c} \prod_{j=1}^s |Z_{c_j}|} \lim_{t \rightarrow 0} \sum_{\lambda \in P_+} \frac{\prod_{j=1}^s \chi_\lambda(c_j)}{d_\lambda^{2g-2+s}} p(\lambda + \rho) e^{-tp_c(\lambda)}.$$

Here N_c is the complex dimension of \mathcal{M}_c and $p(\sqrt{-1}\Omega)$ is a Pontrjagin class of \mathcal{M}_c . By taking derivatives with respect to the c_j 's we can get intersection numbers involving the other generators of the cohomology ring of \mathcal{M}_c , as well as the polynomial property. From index formula we know that the integrals in our formulas contain all the information needed for the famous Verlinde formula. Recently the general Verlinde formula has been directly derived along this line of idea [7].

This localization method of using heat kernels can be applied to other general situation like moment maps, from which we derive the non-abelian localization formula of Witten. See [25] for applications to three dimensional manifolds and see [26] for applications involving finite groups and moment maps.

4. Mirror Principle

Let X be a projective manifold. Let $\mathcal{M}_{g,k}(d, X)$ denote the moduli space of stable maps of genus g and degree d with k marked points into X . Modulo the

obvious equivalence, a point in $\mathcal{M}_{g,k}(d, X)$ is given by a triple $(f; C; x_1, \dots, x_k)$ where $f: C \rightarrow X$ is a degree d holomorphic map and x_1, \dots, x_k are k points on the genus g curve C . Here $d \in H_2(X, \mathbf{Z})$ will also be identified as the integral index (d_1, \dots, d_n) by choosing a basis of $H_2(X, \mathbf{Z})$ dual to a basis of Kahler classes.

This moduli space may have higher dimension than expected. Even worse, its different components may have different dimensions. To define integrals on such space, we need the virtual fundamental cycle first constructed in [19] and later in [6]. Let us denote by $LT_{g,k}(d, X)$ the virtual fundamental cycle which is a homology class of the expected dimension in $\mathcal{M}_{g,k}(d, X)$.

We first consider the case $k = 0$. Let V be a concavex bundle on X . The notion of concavex bundles was introduced in [27], it is a direct sum of a positive and a negative bundle on X . From a concavex bundle V , we can obtain a sequence of vector bundles V_d^g on $M_{g,k}(d, X)$ by taking either $H^0(C, f^*V)$ or $H^1(C, f^*V)$, or their direct sum. Let b be a multiplicative characteristic class. The main problem of mirror principle is to compute the integral [16]

$$K_d^g = \int_{LT_{g,0}(d,X)} b(V_d^g).$$

More precisely, let λ and $T = (T_1, \dots, T_n)$ be formal variables. Mirror principle is to compute the generating series,

$$F(q, \lambda) = \sum_{d, g} K_d^g \lambda^g e^{d \cdot T}$$

in terms of certain natural explicit hypergeometric series. So far we have rather complete picture for the case of balloon manifolds.

A balloon manifold X is a projective manifold with complex torus action and isolated fixed points. Let $H = (H_1, \dots, H_n)$ be a basis of equivariant Kahler classes. Then X is called a balloon manifold if $H(p) \neq H(q)$ when restricted to any two fixed points $p, q \in X$, and the tangent bundle $T_p X$ has linearly independent weights for any fixed point $p \in X$. The complex 1-dimensional orbits in X joining every two fixed points in X are called balloons which are copies of \mathbf{P}^1 . We require the bundle V to have fixed splitting type when restricted to each balloon [27].

Theorem 4.1 ([27]-[30]): *Mirror principle holds for balloon manifolds and concavex bundles.*

In the most interesting cases for the mirror formulas, we simply take characteristic class b to be the Euler class and the genus $g = 0$. The mirror principle implies that mirror formulas actually hold for very general manifolds such as Calabi-Yau complete intersections in toric manifolds and in compact homogeneous manifolds. See [31] and [29] for details. In particular this implies all of the mirror formulas for counting rational curves predicted by string theorists. Actually mirror principle holds even for non-Calabi-Yau and for certain local complete intersections. In [30] we developed the mirror principle for counting higher genus curves, for which the only remaining problem is to find the explicit hypergeometric series. Also our method clearly works well for orbifolds.

As an example, we consider a toric manifold X and genus $g = 0$. Let D_1, \dots, D_N be the toric invariant divisors, and V be the direct sum of line bundles: $V = \bigoplus_j L_j$ with $c_1(L_j) \geq 0$ and $c_1(X) = c_1(V)$. We denote by $\langle \cdot, \cdot \rangle$ the pairing of homology and cohomology classes. Let b be the Euler class and

$$\Phi(T) = \sum_d K_d^0 e^{d \cdot T}$$

where $d \cdot T = d_1 T_1 + \dots + d_n T_n$. Introduce the hypergeometric series

$$HG[B](t) = e^{-H \cdot t} \sum_d \prod_j \prod_{k=0}^{\langle c_1(L_j), d \rangle} (c_1(L_j) - k) \frac{\prod_{\langle D_a, d \rangle < 0} \prod_{k=0}^{-\langle D_a, d \rangle - 1} (D_a + k)}{\prod_{\langle D_a, d \rangle \geq 0} \prod_{k=1}^{\langle D_a, d \rangle} (D_a - k)} e^{d \cdot t}$$

with $t = (t_1, \dots, t_n)$ formal variable.

Corollary 4.2 [29] *There are explicitly computable functions $f(t), g(t) = (g_1(t), \dots, g_n(t))$, such that*

$$\int_X (e^f HG[B](t) - e^{-H \cdot T} e(V)) = 2\Phi - \sum_j T_j \frac{\partial \Phi}{\partial T_j}$$

where $T = t + g(t)$.

From this formula we can determine $\Phi(T)$ uniquely. The functions f and g are given by the expansion of $HG[B](t)$. We can also replace V by general concave bundles [29]. To make our algorithm more explicit, let us consider the the Calabi-Yau quintic, for which we have the famous Candelas formula [9]. In this case $V = \mathcal{O}(5)$ on $X = \mathbf{P}^4$ and the hypergeometric series is:

$$HG[B](t) = e^{Ht} \sum_{d=0}^{\infty} \frac{\prod_{m=0}^{5d} (5H + m)}{\prod_{m=1}^d (H + m)^5} e^{dt},$$

where H is the hyperplane class on \mathbf{P}^4 and t is a parameter. Introduce

$$\mathcal{F}(T) = \frac{5}{6} T^3 + \sum_{d>0} K_d^0 e^{dT}.$$

The algorithm is to take the expansion in H :

$$HG[B](t) = H\{f_0(t) + f_1(t)H + f_2(t)H^2 + f_3(t)H^3\}.$$

Then the famous Candelas formula can be reformulated as

Corollary 4.3: ([27]) *With $T = f_1/f_0$, we have*

$$\mathcal{F}(T) = \frac{5}{2} \left(\frac{f_1}{f_0} \frac{f_2}{f_0} - \frac{f_3}{f_0} \right).$$

Another rather interesting consequence of mirror principle is the local mirror symmetry which is the case when V is a concave bundle. Local mirror symmetry is called geometric engineering in string theory which is used to explain the stringy

origin of the Seiberg-Witten theory. In these cases the hypergeometric series are the periods of elliptic curves which are called the Seiberg-Witten curves. These elliptic curves are the mirror manifolds of the open Calabi-Yau manifolds appeared in the local mirror formulas. For example the total space of the canonical bundle of a del Pezzo surface is an example of open Calabi-Yau manifold covered by the local mirror symmetry. The case \mathbf{P}^2 already has drawn a lot of interests in string theory. The case for $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ on \mathbf{P}^1 easily gives the multiple cover formula.

The key ingredients for the proof of the mirror principle consist of the following: linear and non-linear sigma model, Euler data, balloon and hypergeometric Euler data. As explained in [30], these ingredients are independent of the genus of the curves, except the hypergeometric Euler data, which for $g > 0$ is more difficult to find out, while for the genus 0 case it can be easily read out from localization at the smooth fixed points of the moduli spaces which are covers of the balloons. The interested reader is referred to [27]-[30] for details. Our idea is to go to the equivariant setting and to use the localization formula as given in [1] and its virtual version in [14] on two moduli spaces which we called non-linear and linear sigma models. One key observation is the functorial localization formula [27]-[30]. We apply this formula to the equivariant collapsing map between the two sigma models, and to the evaluation maps. One can see [30] for the existence of the collapsing map for arbitrary genus. Hypergeometric series naturally appear from localizations on the linear sigma models and at the smooth fixed points in the moduli spaces.

Euler data is a very general notion, it can include general Gromov-Witten invariants by adding the pull-back classes by the evaluation map ev_j at the marked points. More precisely we can try to compute integrals of the form:

$$K_{d,k}^g = \int_{LT_{g,k}(d,X)} \prod_j ev_j^* \omega_j \cdot b(V_d^g)$$

where $\omega_j \in H^*(X)$. By introducing the generating series with summation over k , we can still get Euler data. One goal of the most general mirror principle is to explicitly compute such series in terms of hypergeometric series.

We remark that the development of the proof of the mirror formulas owes to many people, first to the string theorists Candelas and his collaborators, Witten, Vafa, Warner, Greene, Morrison, Plesser and many others. They used the physical theory of mirror symmetry, and their computations used mirror manifolds and their periods. For the general theory of mirror principle, see [27]-[30]. See also [13], [12] and [5] for different approaches to the mirror formula.

5. Concluding Remarks.

Localization techniques have been very successful in solving many conjectures from physics. In the meantime string theorists have produced many more exciting new conjectures. We can certainly expect their solutions by using localizations.

Recently several mirror formulas of counting holomorphic discs have been conjectured by Vafa and his collaborators. The boundary of the disc is mapped into a Lagrangian submanifolds of the Calabi-Yau. Other related conjectures include

the Gopakumar-Vafa conjecture on the higher genus multiple covering formula and the mirror formulas for counting higher genus curves in Calabi-Yau. With Chien-Hao Liu we are trying to extend the mirror principle to these settings. Another exciting conjecture is the S-duality conjecture which includes the Witten conjecture on the equality of the Donaldson invariants with the Seiberg-Witten invariants and the Vafa-Witten conjecture on the modularity of the generating series of the Euler numbers of the moduli spaces of self-dual connections. Some progresses are made by constructing a larger moduli space with circle action, the so-called non-abelian monopole moduli spaces. Finally there is the Dijkgraaf-Moore-Verlinde-Verlinde conjecture on the generating series of the elliptic genera of Hilbert schemes. For an approach of using localization, see [34].

References

- [1] M. F. Atiyah & R. Bott, The moment map and equivariant cohomology, *Topology* **23** (1984) 1-28.
- [2] M. F. Atiyah & R. Bott, The Yang-Mills equations over Riemann surfaces, *Phil. Trans. Roy. Soc. London*, **A308**, 523-615 (1982).
- [3] M. F. Atiyah & F. Hirzebruch, Spin manifolds and groups actions, *Essays on topology and Related Topics, Memoirs dédié à Georges de Rham* (ed. A. Haefliger and R. Narasimhan), Springer-Verlag, New York-Berlin (1970), 18-28.
- [4] M. F. Atiyah & I. Singer, The index of elliptic operators. III, *Ann. of Math.* (2) **87** 1968 546-604
- [5] A. Bertram, Another way to enumerate rational curves with torus actions, math.AG/9905159.
- [6] K. Behrend & F. Fentachi, The intrinsic normal cone, *Invent. Math.* **128** (1997) 45-88.
- [7] J.-M. Bismut & F. Labeurie, Symplectic geometry and the Verlinde formulas, *Surveys in differential geometry: differential geometry inspired by string theory, 97-311*, Int. Press, Boston, MA, 1999.
- [8] R. Bott & C. Taubes, On the rigidity theorems of Witten, *J. AMS.* **2** (1989), 137-186.
- [9] P. Candelas, X. de la Ossa, P. Green & L. Parkes, A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory. *Nucl. Phys.* **B359** (1991) 21-74.
- [10] C. Dong, K. Liu & X. Ma, On orbifold elliptic genus, math.DG/0109005.
- [11] C. Dong, K. Liu & X. Ma, Elliptic genus and vertex operator algebras, math.DG/0201135.
- [12] A. Gathmann, Relative Gromov-Witten invariants and the mirror formula, math.AG/0009190.
- [13] A. Givental, Equivariant Gromov-Witten invariants, alg-geom/9603021.
- [14] T. Graber & R. Pandharipande, Localization of virtual classes, alg-geom/9708001.
- [15] F. Hirzebruch, T. Berger & R. Jung, *Manifolds and modular forms*, Aspects of

- Mathematics, E20. Friedr. Vieweg and Sohn, Braunschweig, 1992.
- [16] M. Kontsevich, Enumeration of rational curves via torus actions, hep-th/9405035.
 - [17] P. S. Landweber, *Elliptic Curves and Modular forms in Algebraic Topology*, ed. Landweber P. S., SLNM 1326, Springer, Berlin.
 - [18] B. Lawson & S.-T. Yau, Scalar curvature, non-abelian group actions, and the degree of symmetry of exotic spheres. *Comment. Math. Helv.* **49** (1974), 232–244
 - [19] J. Li & G. Tian, Virtual moduli cycle and Gromov-Witten invariants of algebraic varieties, *J. of Amer. math. Soc.* **11**, no. 1, (1998) 119-174.
 - [20] K. Liu, On $SL_2(\mathbf{Z})$ and topology, *Math. Research Letter*, **1** (1994), 53-64.
 - [21] K. Liu, On Modular invariance and rigidity theorems, *J. Diff. Geom.* **41** (1995), 343-396.
 - [22] K. Liu, Modular invariance and characteristic numbers, *Comm. Math. Phys.*, **174** (1995), no. 1, 29-42.
 - [23] K. Liu & W. Zhang, Elliptic genus and η -invariant, *International Math. Res. Notices*, no. 8 (1994), 319-327.
 - [24] K. Liu, Heat kernel and moduli space, *Math. Res. Letter*, **3** (1996), 743-762.
 - [25] K. Liu, Heat kernel and moduli space II, *Math. Res. Letter*, **4** (1997), 569-588.
 - [26] K. Liu, Heat kernels, symplectic geometry, moduli spaces and finite groups, *Surveys in differential geometry: differential geometry inspired by string theory*, 527–542, Int. Press, Boston, MA, 1999
 - [27] B. Lian, K. Liu & S.-T. Yau, Mirror principle, I, *Asian J. Math.* **1** (1997), pp. 729 - 763;
 - [28] B. Lian, K. Liu & S.-T. Yau, Mirror principle, II, **3** (1999), no. 1, 109–146
 - [29] B. Lian, K. Liu & S.-T. Yau, Mirror principle, III, *Asian J. Math.* **3** (1999), no. 4, 771–800.
 - [30] B. Lian, K. Liu & S.-T. Yau, Mirror principle, IV, *Asian J. Math.*
 - [31] B. Lian, C.-H. Liu, K. Liu & S.-T. Yau, The S^1 fixed points in Quot-schemes and mirror principle computations, math.AG/0111256.
 - [32] K. Liu, X. Ma & W. Zhang, On elliptic genera and foliations, *Math. Res. Lett.*, **8** (2001), no. 3, 361–376
 - [33] K. Liu & X. Ma, On family rigidity theorems I. *Duke Math. J.* **102** (2000), 451-474.
 - [34] K. Liu, X. Ma & J. Zhou, The elliptic genus of the Hilbert schemes of surfaces, work in progress.
 - [35] C. Taubes, S^1 -actions and elliptic genera, *Comm. Math. Phys.* **122** (1989), 455-526.
 - [36] E. Witten, The index of the Dirac operator in loop space, in *Elliptic Curves and Modular forms in Algebraic Topology*, Landweber P.S., SLNM 1326, Springer, Berlin, 161-186.
 - [37] E. Witten, On quantum gauge theory in two dimensions, *Commun. Math. Phys.* **141** (1991), 153-209.
 - [38] E. Witten, Two-dimensional gauge theory revisited, *J. Geom. Phys.* **9** (1992), 303-368.