Elliptic Genus and

Kac-Moody Algebras

Modular Invariance in Topology:

Rigidity, Vanishing, Divisibility.

Infinite Dimension $\Rightarrow$ finite dimension.

Algebra $\Rightarrow$ Geometry and Topology.

M, compact smooth spin manifold;

LM, its loop space

$=$ smooth maps from S^1 into M

$M \hookrightarrow LM$ as fixed points of rotation.

Elliptic genus:

Index theory on loop space LM.

Origin:

Rigidity $+$ **Physics**.

Obstruction to compact Lie group action on manifolds; anomaly cancellation in physics.

M compact smooth manifold with S^1-action; P an equivariant elliptic operator. For $g \in S^1$, define Lefschetz number:

$$F_P(g) = \mathrm{tr}_g \mathrm{Ker}\, P - \mathrm{tr}_g \mathrm{Coker}\, P \in \mathbf{R}(S^1).$$

P is **rigid** $\Leftrightarrow F_P(g)$ is constant.

Strong: P has vanishing property:

$F_P(g) = 0 \Rightarrow \mathrm{Ind}\, P = 0.$

History of the subject:

Trivial: de Rahm operator d. Homotopy invariant.

1970: Atiyah-Hirzebruch, (Luztig, Kosniwski): the Dirac operator D, signature operator d_S, $\bar{\partial}$-operator. (Atiyah: most surprising application of fixed point formula).

1982: Witten, $D \otimes TM$ is rigid for homogeneous spin M. (Kaluza-Klein theory).

1986: Landweber-Stong, Ochanine, Witten, Lerche-Warner.... Elliptic genera. (Cobordism for semi-free action, multiplicative for fibration. Chudonovsky's modular functions....)

LSO-Ellipitc genus = signature of loop space.

More (Witten): Loop space $\widehat{A}$-genus, twisted by wedge operations of bundles.... (actually from level 1 loop group representations).

Another characterization: Multiplicative for spin fibration.

(Signature: multiplicative for orientable fibration).

New genus from more special fibration?

(Formally) Atiyah-Bott fixed point formula on loop space gives the following twisted elliptic operators:

(1). $d_S \otimes \bigotimes_{n=1} S_{q^n}(TM) \otimes \bigotimes_{m=1} \bigwedge_{q^m}(TM)$;

(2). $D \otimes \bigotimes_{n=1} S_{q^n}(TM) \otimes \bigotimes_{m=1} \bigwedge_{q^{m-\frac{1}{2}}}(TM)$;

(3). $D \otimes \bigotimes_{n=1} S_{q^n}(TM) \otimes \bigotimes_{m=1} \bigwedge_{-q^{m-\frac{1}{2}}}(TM)$.

Witten conjectures: These elliptic operators are rigid.

Remark:

(1). Each operator contains infinitely many terms: expand in q series, say (2):

$$D + D \otimes TM\, q^{\frac{1}{2}} + \cdots .$$

infinitely many operators.

(2). $d_S = D \otimes \triangle(M)$, so is every elliptic operator on a spin manifold. Dirac operator

$$D: \ \Gamma(M, S^+) \to \Gamma(M, S^-)$$

generates everything. S^+, S^- half spin representations. $\triangle(M) = S^+ + S^-$.

Index formula $\Rightarrow$ the indices are topological invariants.

(3). Operations:

$S_t(V) = 1 + tV + t^2 S^2(V) + \cdots$;

$\wedge_t(V) = 1 + tV + t^2 \wedge^2(V) + \cdots$.

Even the rigidity of $D \otimes TM$ has **no** direct proof without using the rigidity of elliptic genus.

Solution to Witten conjectures (1986-1989):

Taubes, Bott-Taubes. (Complex analogue: Hirzebruch, Krichever, 1989.)

Techniques: "Transfer argument", working on one operator each time. detailed analysis of the topology and combinatorics of fixed points. Fredholm analysis (Taubes).

Modular invariance (1992-1993, Liu): Simple proofs of the conjectures, working on 3 operator together. Modular transformations interchange them.

Discover general rigidity involving loop group representations, vanishing theorems, divisibility theorems.

More!?

Formal comparison of results:

$\hat{A}$-vanishing $\leftrightarrow$ Witten genus vanishing.

Signature rigid $\leftrightarrow$ Elliptic genus rigid.

? $\leftrightarrow$ Loop group representations.

? $\leftrightarrow$ Elliptic theorems.

? $\leftrightarrow$ Miraculous cancellation.

Some results derived by using modular invariance can not be seen from finite dimension, like the elliptic vanishing theorems.

General rigidity theorem from loop group representations.

$\tilde{L}$Spin$(2l)$, (central extension) of loop group; E, a positive energy representation.

$$E = \bigoplus_{n=0}^{\infty} E_n$$

under rotation action by S^1.

V, Spin$(2l)$ equivariant vector bundle on M, P, its frame bundle.

$\tilde{E}_n = P \times_{\text{Spin}(2l)} E_n$ associated bundle.

Write generating series:

$$\psi(E, V) = \sum_{n=0}^{\infty} \tilde{E}_n \, q^n$$

with $q = e^{2\pi i\tau}$ formal parameter.

Let $p_1(V)_{S^1}$ denote the first equivariant Pontrjagin class.

Theorem: For every positive energy representation E of highest weight of level m, if $p_1(M)_{S^1} = m\, p_1(V)_{S^1}$, then

$$D \otimes \bigotimes_{n=1} S_{q^n}(TM) \otimes \psi(E, V)$$

is rigid.

Remark:

(1). Level $m = 1$ and $V = TM \Rightarrow$ Witten rigidity conjectures, (and the complex analogue).

(2). If $mp_1(V)_{S^1} - p_1(M)_{S^1} = n\, \pi^* u^2$, then the Lefschetz number is a holomorphic Jacobi form. If $n < 0$, then it is $\equiv 0$.

Here

$$\pi : \; M \times_{S^1} ES^1 \to BS^1$$

is the natural projection; $u \in H^*(BS^1, \mathbb{Z}) = \mathbb{Z}[[u]]$.

$p_1(V)_{S^1} \in H^4(M \times_{S^1} ES^1, \mathbb{Z})$, equivariant cohomology group.

(3). Rigidity associated to general loop group representations. (Gong-Liu).

Idea of Proof: Assume isolated fixed points $\{P\}$. Apply the Atiyah-Bott-Segal-Singer fixed point formula, we get (up to normalization):

$$F_P(t) = \sum_P H(t, \tau)\, c_E(t, \tau)$$

where

$$H(t,\tau) = \prod_j \frac{\theta'(0,\tau)}{\theta(m_j t,\tau)}$$

and

$$c_E(t,\tau) = \chi_E(T,\tau)$$

with $T = (n_1 t, \cdots, n_l t)$. Here m_i, n_j weights of TM and V at p, from S^1-action. χ_E the Kac-Weyl character of the representation E.

Jacobi-theta function:

$$\theta(t,\tau) = c\, q^{\frac{1}{4}} \sin\frac{t}{2} \prod_{j=1}^{\infty} (1 - q^j e^{2\pi i t})(1 - q^j e^{-2\pi i t}).$$

Rigidity $\Leftrightarrow$ $F_P(t)$ is independent of t.

But looks like a lot of poles!

Magic of geometry: they all cancel.

Step 1: Index theory $\Rightarrow$ no pole on S^1, or $t \in \mathbb{R}$.

Step 2: Symmetry under $SL(2, \mathbf{Z}) \Rightarrow$ no pole for any $t \in \mathbb{C}$. Modularity of Kac-Weyl character formula (Kac-Peterson).

Step 3: Elliptic with respect to $t \Rightarrow$ constant.

Remark: Combination of modularity and index theory.

It is slightly subtler to prove the vanishing theorems: basic properties of holomorphic Jacobi forms.

Most interesting and surprising: Loop space analogue of the Atiyah-Hirzebruch theorem:

Theorem: If $p_1(M)_{S^1} = n\,\pi^* u^2$ for any integer n, then the index of

$$D_L = D \otimes \bigotimes_{n=1} S_{q^n}(TM)$$

vanishes.

If S^1-action is induced from and S^3-action, then $p_1(M)_{S^1} = n\,\pi^* u^2 \Leftrightarrow p_1(M) = 0$.

Corollary: If there exists S^3-action, and $p_1(M) = 0$, then the index of

$$D_L = D \otimes \bigotimes_{n=1} S_{q^n}(TM)$$

vanishes.

This operator is "the Dirac operator" of loop space, so most basic. Other operators are from its twists.

Hoehn-Stolz conjecture: positive Ricci curvature and $p_1(M) = 0$ implies vanishing index.

(Loop space analogue of Lichnerovicz theorem for positive scalar curvature).

Another pleasant corollary (of the method):

Miraculous cancellation formula: express the L-form in terms of the twisted $\widehat{A}$-forms. (12-dimension, Alveraz-Gaume, Witten).

Special case (Ochanine): For a dimension $8k+4$ spin manifold M,

$$\mathrm{Sign}(M) \equiv 0\,(16).$$

Recent works.

Family version: Joint with X. Ma and W. Zhang.

Let $p : M \to B$ a fibration with compact fiber X, with fiberwise S^1-action. D^X fiberwise Dirac operator. V, an equivariant Spin($2l$) vector bundle on M.

Family of elliptic operators:

$$D^X \otimes \bigotimes_{n=1} S_{q^n}(TX) \otimes \psi(E, V)$$

with equivariant index

$$F_P(g) \in K_{S^1}(B).$$

Theorem: If $p_1(M)_{S^1} = m\, p_1(V)_{S^1}$, then the family elliptic operator

$$D^X \otimes \bigotimes_{n=1} S_{q^n}(TX) \otimes \psi(E,V)$$

is rigid on the Chern character level.

Remark:

(1). Rigid on Chern character level $\simeq$ ch$(F_P(g)) \in H^*_{S^1}(B)$ is independend of $g \in S^1$. Many lower degree terms are rigid.

(2). Level $m = 1 \Rightarrow$ family Witten rigidity theorems.

(3). K-theory version, without taking Chern character, but only for level 1.

Foliated version: Joint with Ma nd Zhang. The action preserves the leaves on a foliated manifold M. If the leaves are spin, then similar rigidity and vanishing theorems.

M needs not be spin. (Again modularity)

Orbifold version: Joint with C. Dong and Ma. For an orbifold M with S^1-action.

Define orbifold elliptic genus: keep the modularity.

(4). Interests in algebraic geometry. Singular elliptic genus (Borisov-Libgober, Wang, Totaro, ...) also has rigidity property for Calabi-Yau manifolds:

Elliptic genera (for complex manifolds) give the only characteristic numbers that may be defined independent of the resolution of singularities. (Rigidity used)

Elliptic genus for (almost) complex manifold X: $TX \otimes \mathbb{C} = T'X \oplus T''X$:

$\bar{\partial} \otimes \bigotimes_{n=0} \bigwedge_{-y^{-1}q^n}(T''X) \otimes \bigotimes_{n=1} \bigwedge_{-yq^n}(T'X) \otimes \bigotimes_{n=1} S_{q^n}(TX \otimes \mathbb{C})$

where $y \in \mathbb{C}$ a parameter.

Theorem: (i) If $c_1(X) \equiv 0\,(N)$ and y an N-th root of unity, then this operator is rigid.

(ii) If X is Calabi-Yau ($c_1(X) = 0$), this operator is rigid for **any** y.

This is a special case of a general elliptic operator involving complex vector bundles, or general loop group representations.

Proof is the same: modularity + index theory.

Have been trying to get in touch with

Vertex operator algebras: Bundles of VOA can be constructed without problem.

Missing: Modularity.

Good algebraic results will have applications in geometry and topology sooner or later. Virasoro algebra?