Geometric Aspects of the Moduli Spaces of Riemann Surfaces

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Moduli spaces and Teichmüller spaces of Riemann surfaces have been studied for many many years, since Riemann.

Ahlfors, Deligne, Mumford, Yau, Witten, Kontsevich, McMullen....

They have appeared in many subjects of mathematics, from geometry, topology, algebraic geometry, to number theory: Faltings' proof of the Mordell conjecture.

They have also appeared in theoretical physics like string theory.

Many computations of path integrals are reduced to integrals of Chern classes on such moduli spaces.

We will consider the cases of genus $g \geq 2$, the geometry and topological aspects.

The Teichmüller space $\mathcal{T}_g$ is a domain of holomorphy embedded in $\mathbb{C}^n$ with $n = 3g - 3$. The moduli space $\mathcal{M}_g$ is an orbifold, as a quotient of $\mathcal{T}_g$ by mapping class group.

The topology of Teichmüller space is trivial, but the topology of the moduli space and its compactification have highly nontrivial topology, and have been well-studied for the past years:

Example: Hodge integrals: Witten conjecture(Kontsevich); Marino-Vafa conjecture (Liu-Liu-Zhou), both came from string theory.

Marino-Vafa conjecture we proved gives a closed formula for the generating series of triple Hodge integrals of all genera and all possible marked points, in terms of Chern-Simons knot invariants.

Hodge integrals are just the intersection numbers of λ classes and ψ classes on the Deligne-Mumford moduli space of stable Riemann surfaces $\overline{\mathcal{M}}_{g,h}$, the moduli with h marked points.

A point in $\overline{\mathcal{M}}_{g,h}$ consists of $(C, x_1, \ldots, x_h)$, a (nodal) Riemann surface and h smooth points on C.

The Hodge bundle $\mathbb{E}$ is a rank g vector bundle over $\overline{\mathcal{M}}_{g,h}$ whose fiber over $[(C, x_1, \ldots, x_h)]$ is $H^0(C, \omega_C)$. The λ classes are Chern Classes:

$$\lambda_i = c_i(\mathbb{E}) \in H^{2i}(\overline{\mathcal{M}}_{g,h}; \mathbb{Q}).$$

The cotangent line $T^*_{x_i}C$ of C at the i-th marked point x_i gives a line bundle $\mathbb{L}_i$ over $\overline{\mathcal{M}}_{g,h}$. The ψ classes are also Chern classes:

$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,h}; \mathbb{Q}).$$

Define

$$\Lambda_g^\vee(u) = u^g - \lambda_1 u^{g-1} + \cdots + (-1)^g \lambda_g.$$

Mariño-Vafa conjecture:
Generating series of triple Hodge integrals

$$\int_{\overline{\mathcal{M}}_{g,h}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(\tau)\Lambda_g^\vee(-\tau-1)}{\prod_{i=1}^h (1-\mu_i\psi_i)},$$

for all g and all h can be expressed by close formulas of *finite* expression in terms of representations of symmetric groups, or Chern-Simons knot invariants. Here τ is a parameter, μ_i some integers.

Conjectured from large N duality between Chern-Simons and string theory. Proved by using differential equations from both geometry and combinatorics.

Example: The Mumford conjecture about generators of the stable cohomology of the moduli spaces. This was proved recently by Madsen et al.

Our current project is to study the geometry of the Teichmüller and the moduli spaces. More precisely to understand the various metrics on these spaces, and more important their applications. Also introduce new metrics with good property. The results are contained in

1. *Canonical Metrics in the Moduli Spaces of Riemann Surfaces I*, math.DG/0403068.

2. *Canonical Metrics in the Moduli Spaces of Riemann Surfaces II*. math.DG/0409220.

by K. Liu, X. Sun, S.-T. Yau.

The key point is the understanding of the Ricci and the perturbed Ricci metric: two new complete Kähler metrics. Their curvatures and boundary behaviors are studied in details.

Interesting applications to the geometry, like the stability of the logarithmic cotangent bundle of moduli spaces and more will follow.

There are many very famous classical metrics on the Teichmüller and the moduli spaces:

(1). **Finsler metrics:**

Teichmüller metric;
Caratheodory metric; Kobayashi metric.

(2). **Kähler metrics:**

The Weil-Petersson metric, (Incomplete).
Cheng-Yau-Mok's Kähler-Einstein metric; McMullen metric; Bergman metric; Asymptotic Poincare metric.

Ricci metric and perturbed Ricci metric.

The above seven metrics are complete Kähler metrics.

Selected applications of these metrics:

Example: Royden proved that

Teichmüller metric = Kobayashi metric.

This implies that the isometry group of $\mathcal{T}_g$ is exactly the mapping class group.

Example: Ahlfors: the Weil-Petersson (WP) metric is Kähler, the holomorphic sectional curvature is negative.

Masur: WP metric is incomplete.

Wolpert studied WP metric in great details, found many important applications in topology(relation to Thurston's work) and algebraic geometry(relation to Mumford's work).

Each family of semi-stable curves induces a holomorphic maps into the moduli space. By applying the Schwarz-Yau lemma to this map, we immediately get very *sharp* geometric height inequalities in algebraic geometry.

Corollaries include:

1. Kodaira surface X has strict Chern number inequality: $c_1(X)^2 < 3c_2(X)$.

2. Beauville conjecture: the number of singular fibers for a non-isotrivial family of semi-stable curves over $\mathbb{P}^1$ is at least 5.

Geometric Height Inequalities, by Kefeng Liu, MRL 1996.

Finiteness results for families of projective manifolds, like the Mordell type conjecture, follow from such height inequalities, so from the negative curvature properties of moduli spaces.

Example: McMullen proved that the moduli spaces of Riemann surfaces are Kähler hyperbolic, by using his own metric which he obtained by perturbing the WP metric.

This means bounded geometry and the Kähler form on the Teichmüller space is of the form $d\alpha$ with α bounded one form.

This gives interesting corollaries on the geometry and the topology of the moduli and the Teichmüller spaces:

The lowest eigenvalue of the Laplacian on the Teichmüller space is positive.

Only middle dimensional L^2 cohomology is nonzero on the Teichmüller space.

Our Goal:
(1) To understand the known complete metrics on the moduli space and Teichmüller space, and their geometry, and introduce new metics.

Most interesting: the two new complete Kähler metrics, the **Ricci metric** and the **Perturbed Ricci metric**. We have rather complete understanding of these two new metrics.

(2) With the help of the new metrics we have much better understanding of the Kähler-Einstein metric: boundary behavior.

The curvature of KE metric and all of its covariant derivatives are bounded on the Teichmüller space: bounded geometry.

(3) Algebro-geometric consequences: the log cotangent bundle of the DM moduli space of stable curves is stable.

The DM moduli $\overline{\mathcal{M}}_g$ is of log general type....

Conventions:

For a Kähler manifold (M^n, g) with local holomorphic coordinates $z_1, \cdots, z_n$, the curvature of g is given by

$$R_{i\bar{j}k\bar{l}} = \frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} - g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z_k} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}_l}.$$

In this case, the Ricci curvature is

$$R_{i\bar{j}} = -g^{k\bar{l}} R_{i\bar{j}k\bar{l}} = -\partial_{z_i} \partial_{\bar{z}_j} \log \det(g_{i\bar{j}}).$$

The holomorphic sectional curvature is negative means

$$R(v, \bar{v}, v, \bar{v}) > 0.$$

Two Kähler metrics g_1 and g_2 are equivalent or two norms $\| \cdot \|_1$ and $\| \cdot \|_2$ are equivalent if there is a constant $C > 0$ such that

$$C^{-1} g_1 \leq g_2 \leq C g_1$$

or

$$C^{-1} \| \cdot \|_1 \leq \| \cdot \|_2 \leq C \| \cdot \|_1.$$

We denote this by $g_1 \sim g_2$ or $\| \cdot \|_1 \sim \| \cdot \|_2$.

Basics of the Teichmüller and the Moduli Spaces:

Fix an orientable surface Σ of genus $g \geq 2$.

• *Uniformization Theorem.*
Each Riemann surface of genus $g \geq 2$ can be viewed as a quotient of the hyperbolic plane $\mathbb{H}^2$ by a Fuchsian group. Thus there is a unique KE metric, or the hyperbolic metric on Σ.

The group $Diff^+(\Sigma)$ of orientation preserving diffeomorphisms acts on the space $\mathcal{C}$ of all complex structures on Σ by pull-back.

• *Teichmüller space.*

$$\mathcal{T}_g = \mathcal{C}/Diff_0^+(\Sigma)$$

where $Diff_0^+(\Sigma)$ is the set of orientation preserving diffeomorphisms which are isotopic to identity.

• *Moduli space.*

$$\mathcal{M}_g = \mathcal{C}/Diff^+(\Sigma) = \mathcal{T}_g/Mod(\Sigma)$$

is the quotient of the Teichmüller space by the mapping class group where

$$Mod\,(\Sigma) = Diff^+(\Sigma)/Diff_0^+(\Sigma).$$

• *Dimension.*

$$\dim_{\mathbb{C}} \mathcal{T}_g = \dim_{\mathbb{C}} \mathcal{M}_g = 3g - 3.$$

$\mathcal{T}_g$ is a pseudoconvex domain in $\mathbb{C}^{3g-3}$. $\mathcal{M}_g$ is a complex orbifold.

• *Tangent and cotangent space.*
By the deformation theory of Kodaira-Spencer and the Hodge theory, for any point $X \in \mathcal{M}_g$,

$$T_X\mathcal{M}_g \cong H^1(X, T_X) = HB(X)$$

where $HB(X)$ is the space of harmonic Beltrami differentials on X.

$$T_X^*\mathcal{M}_g \cong Q(X)$$

where $Q(X)$ is the space of holomorphic quadratic differentials on X.

For $\mu \in HB(X)$ and $\phi \in Q(X)$, the duality between $T_X\mathcal{M}_g$ and $T_X^*\mathcal{M}_g$ is

$$[\mu : \phi] = \int_X \mu\phi.$$

Teichmüller metric is the L^1 norm. The WP metric is the L^2 norm.

The following theorem is an easy corollary of our study of the new metrics:

Theorem. *On the moduli space $\mathcal{M}_g$, the Teichmüller metric $\|\cdot\|_T$, the Kobayashi metric, the Caratheodory metric $\|\cdot\|_C$, the Kähler-Einstein metric ω_{KE}, the McMullen metric ω_M, the asymptotic Poincaré metric ω_P the Ricci metric ω_τ and the perturbed Ricci metric $\omega_{\tilde{\tau}}$ are equivalent. Namely*

$$\omega_{KE} \sim \omega_{\tilde{\tau}} \sim \omega_\tau \sim \omega_P \sim \omega_M$$

and

$$\|\cdot\|_K = \|\cdot\|_T \sim \|\cdot\|_C \sim \|\cdot\|_M.$$

On Teichmüller space $\mathcal{T}_g$, the Bergman metric ω_B is also equivalent to the above metrics.

We therefore have proved the following conjecture of Yau made in the early 80s:

Theorem. *The Kähler-Einstein metric is equivalent to the Teichmüller metric on the moduli space:* $\|\cdot\|_{KE} \sim \|\cdot\|_T$.

Another corollary was also conjectured by Yau as one of his 120 famous problems:

Theorem. *The Kähler-Einstein metric is equivalent to the Bergman metric on the Teichmüller space:* $\omega_{KE} \sim \omega_B$.

Remark(Wolpert): The question on relation between Carathéodory metric and the Bergman metric was raised by Bers in the early 70s.

KE Metric and Consequences: The detailed understanding of the boundary behaviors of these metrics gives us geometric corollaries:

• Stability of the logarithmic cotangent bundle of the DM moduli spaces.

Let $\bar{E}$ denote the logarithmic extension of the cotangent bundle of $\mathcal{M}_g$. Recall that it consists of sections of the form:

$$\sum_{i=1}^{m} a_i(t,s)\frac{dt_i}{t_i} + \sum_{i=m+1}^{n} a_i(t,s)ds_i,$$

where recall that the compactification divisor is defined by $\prod_{i=1}^{m} t_i = 0$, with $n = 3g - 3$.

Theorem: *The first Chern class $c_1(\bar{E})$ is positive and $\bar{E}$ is stable with respect to $c_1(\bar{E})$.*

Remarks:

(1). This means for any sub-bundle F of $\bar{E}$, we have

$$\frac{\deg(F)}{\operatorname{rank}(F)} < \frac{\deg(\bar{E})}{\operatorname{rank}(\bar{E})}$$

where the degree is with respect to $c_1(\bar{E})$:

$$\deg(F) = \int_{\overline{\mathcal{M}}_g} c_1(F) c_1(\bar{E})^{n-1}.$$

(2). The proof is an application of the existence and the precise boundary behavior of the KE metric, to make sure the degrees well-defined.

(3). Corollary: the DM moduli space is of logarithmic general type for $g \geq 2$. Mumford (1977) proved that, $\overline{\mathcal{M}}_{g,h}$ is log general type for $h \geq 3$.

• **Bounded geometry:**

Theorem: *The curvature of the KE metric and all of its covariant derivatives are uniformly bounded on the Teichmüller spaces, and its injectivity radius has lower bound.*

As corollary we have

Theorem: *The Ricci and the perturbed Ricci metric also have bounded geometry.*

We hope to find a complete Kähler metric on $\mathcal{M}_g$ whose holomorphic sectional curvature is bounded above by a negative constant and whose Ricci curvature is bounded from below by a constant. Best possible metric is a complete Kähler metric with non-positive Riemannian bisectional curvature and bounded geometry .

Observation:
Since the Ricci curvature of the Weil-Petersson metric is bounded above by a negative constant, one can use the negative Ricci curvature of the WP metric to define a new metric.

We call this metric the **Ricci metric**

$$\tau_{i\bar{j}} = -Ric(\omega_{WP})_{i\bar{j}}.$$

It turns out that the curvature properties of the Ricci metric is not good enough for our purpose.

New Idea:
Perturbed the Ricci metric with a large constant multiple of the WP metric. We define the **perturbed Ricci metric**

$$\omega_{\tilde{\tau}} = \omega_{\tau} + C\,\omega_{WP}.$$

The perturbed Ricci metric has some desired good curvature properties.

Theorem. *Let τ be the Ricci metric on the moduli space $\mathcal{M}_g$. Then*

- *τ is equivalent to the asymptotic Poincaré metric.*
- *The holomorphic sectional curvature of τ is asymptotically negative in the degeneration directions.*
- *The holomorphic sectional curvature, the bisectional curvature and the Ricci curvature of τ are bounded.*

We can explicitly write down the asymptotic behavior of this metric: asymptotic Poincaré:

$$\sum_{i=1}^{m} \frac{C_i \, |dt_i|^2}{|t_i|^2 \log^2 |t_i|} + \sum_{i=m+1}^{n} ds_i^2.$$

t_i's are the coordinates in the degeneration directions: $t_i = 0$ define the divisor.

To get control on the signs of the curvatures, we need to perturb the Ricci metric. Recall that the curvatures of the WP metric are negative.

Theorem. *Let* $\omega_{\tilde{\tau}} = \omega_\tau + C\omega_{WP}$ *be the perturbed Ricci metric on* $\mathcal{M}_g$*. Then for suitable choice of the constant* C*, we have*
- $\tilde{\tau}$ *is a complete Kähler metric with finite volume.*
- *The holomorphic sectional curvature and the Ricci curvature of* $\tilde{\tau}$ *are bounded from above and below by negative constants.*

Remark:
- The perturbed Ricci metric is the first known complete Kähler metric on the moduli space whose holomorphic sectional and Ricci curvature have negative bounds, and bounded geometry. It is close to be the best possible metric on the moduli and the Teichmüller space.

Sketch of the Proofs of Equivalences:
This is easy from our understanding of the new metrics, together with Yau's Schwarz lemma.

- $\|\cdot\|_T = \|\cdot\|_K$ was proved by Royden.

- $\|\cdot\|_T \sim \|\cdot\|_M$ was proved by McMullen.

- $\omega_P \sim \omega_\tau$ is proved by comparing their asymptotic behavior.

- $\omega_\tau \sim \omega_{\tilde{\tau}}$:

Consider the identity map:

$$id : (\mathcal{M}_g, \omega_\tau) \to (\mathcal{M}_g, \omega_{WP}).$$

Yau's Schwarz Lemma $\Rightarrow \omega_{WP} \leq C_0 \omega_\tau$. So

$$\omega_\tau \leq \omega_{\tilde{\tau}} = \omega_\tau + C\omega_{WP} \leq (CC_0 + 1)\omega_\tau.$$

- $\omega_{\tilde{\tau}} \sim \omega_{KE}$:

Consider the identity map:

$$id : (\mathcal{M}_g, \omega_{KE}) \to (\mathcal{M}_g, \omega_{\tilde{\tau}})$$

and

$$id : (\mathcal{M}_g, \omega_{\tilde{\tau}}) \to (\mathcal{M}_g, \omega_{KE}).$$

Yau's Schwarz Lemma $\Rightarrow$

$$\omega_{\tilde{\tau}} \leq C_0 \omega_{KE}$$

and

$$\omega_{KE}^n \leq C_0 \omega_{\tilde{\tau}}^n.$$

The equivalence follows from linear algebra.

• $\omega_M \sim \omega_{\tilde{\tau}}$:
Consider the identity map

$$id : (\mathcal{M}_g, \omega_M) \to (\mathcal{M}_g, \omega_{\tilde{\tau}}).$$

Yau's Schwarz Lemma $\Rightarrow$

$$\omega_{\tilde{\tau}} \leq C_0 \omega_M.$$

The other side is proved by asymptotic analysis and linear algebra.

The equivalence $\| \cdot \|_C \sim \| \cdot \|_K \sim \| \cdot \|_B$ on Teichmüller space is proved by using the Bers embedding and the basic properties of these metrics.

Curvature formulas:

- *Weil-Petersson metric*

Let $\mathfrak{X}$ be the total space over the $\mathcal{M}_g$ and π be the projection map.

Pick $s \in \mathcal{M}_g$, let $\pi^{-1}(s) = X_s$. Let $s_1, \cdots, s_n$ be local holomorphic coordinates on $\mathcal{M}_g$ and s and let z be local holomorphic coordinate on X_s.

Recall

$$T_s\mathcal{M}_g \cong HB(X_s).$$

The Kodaira-Spencer map is

$$\frac{\partial}{\partial s_i} \mapsto A_i \frac{\partial}{\partial z} \otimes d\bar{z} \in HB(X_s).$$

The Weil-Petersson metric is

$$h_{i\bar{j}} = \int_{X_s} A_i \bar{A}_j \ dv$$

where $dv = \frac{\sqrt{-1}}{2}\lambda dz \wedge d\bar{z}$ is the volume form of the KE metric λ on X_s.

By the work of Siu and Schumacher, let

$$a_i = -\lambda^{-1}\partial_{s_i}\partial_{\bar{z}} \log \lambda.$$

Then

$$A_i = \partial_{\bar{z}} a_i.$$

Let η be a relative $(1,1)$ form on $\mathfrak{X}$. Then

$$\frac{\partial}{\partial s_i}\int_{X_s} \eta = \int_{X_s} L_{v_i}\eta$$

where

$$v_i = \frac{\partial}{\partial s_i} + a_i \frac{\partial}{\partial z}$$

is called the harmonic lift of $\frac{\partial}{\partial s_i}$.

In the following, we let

$$f_{i\bar{j}} = A_i \bar{A}_j \text{ and } e_{i\bar{j}} = T(f_{i\bar{j}}).$$

Here $T = (\Box + 1)^{-1}$ is the Green operator. The functions $f_{i\bar{j}}$ and $e_{i\bar{j}}$ will be the building blocks of the curvature formula.

• *Curvature formula of the WP metric.*
By the work of Wolpert, Siu and Schumacher, the curvature of the Weil-Petersson metric is

$$R_{i\bar{j}k\bar{l}} = \int_{X_s} (e_{i\bar{j}} f_{k\bar{l}} + e_{i\bar{l}} f_{k\bar{j}}) \; dv.$$

Remark:
The sign of the curvature of the WP metric can be seen directly.

The precise upper bound $-\frac{1}{2\pi(g-1)}$ of the holomorphic sectional curvature and the Ricci curvature of the WP metric can be obtained by spectrum decomposition of the operator $(\Box + 1)$.

The curvature of the WP metric is not bounded from below. But *surprisingly* the Ricci and the perturbed Ricci have bounded curvatures.

• *Curvature formula of the Ricci metric*

The curvature $\widetilde{R}_{i\bar{j}k\bar{l}}$ of the Ricci metric is

$$\begin{aligned}\widetilde{R}_{i\bar{j}k\bar{l}} =& h^{\alpha\bar{\beta}}\left\{\sigma_1\sigma_2\int_{X_s} T(\xi_k(e_{i\bar{j}}))\bar{\xi}_l(e_{\alpha\bar{\beta}})\ dv\right\} \\ & + h^{\alpha\bar{\beta}}\left\{\sigma_1\sigma_2\int_{X_s} T(\xi_k(e_{i\bar{j}}))\bar{\xi}_\beta(e_{\alpha\bar{l}})\ dv\right\} \\ & + h^{\alpha\bar{\beta}}\left\{\sigma_1\int_{X_s} Q_{k\bar{l}}(e_{i\bar{j}})e_{\alpha\bar{\beta}}\ dv\right\} \\ & - \tau^{p\bar{q}}h^{\alpha\bar{\beta}}h^{\gamma\bar{\delta}}\left\{\sigma_1\int_{X_s}\xi_k(e_{i\bar{q}})e_{\alpha\bar{\beta}}\ dv\right\}\times \\ & \left\{\widetilde{\sigma}_1\int_{X_s}\bar{\xi}_l(e_{p\bar{j}})e_{\gamma\bar{\delta}})\ dv\right\} \\ & + \tau_{p\bar{j}}h^{p\bar{q}}R_{i\bar{q}k\bar{l}}.\end{aligned}$$

Here σ_1 is the symmetrization of indices i, k, α.
σ_2 is the symmetrization of indices j, β.
$\widetilde{\sigma}_1$ is the symmetrization of indices j, l, δ.
ξ_k and $Q_{k\bar{l}}$ are combinations of the Maass operators and the Green operators.

Too complicated to see the sign. We work out its asymptotics near the boundary.

Asymptotics:

• *Deligne-Mumford Compactification*
For a Riemann surface X, a point $p \in X$ is a node if there is a neighborhood of p which is isomorphic to the germ

$$\{(u,v) \mid uv = 0, |u| < 1, |v| < 1\} \subset \mathbb{C}^2.$$

A Riemann surface with nodes is called a nodal surface.

A nodal Riemann surface is stable if each connected component of the surface subtract the nodes has negative Euler characteristic. In this case, each connected component has a complete hyperbolic metric.

The union of $\mathcal{M}_g$ and stable curves of genus g is the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$, the DM moduli.

$D = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ is a divisor of normal crossings.

• *Principle*
To compute the asymptotics of the Ricci metric and its curvature, we work on surfaces near the boundary of $\mathcal{M}_g$. The geometry of these surfaces localize on the pinching collars.

• *Model degeneration* (Wolpert)
Consider the variety

$$V = \{(z, w, t) \mid zw = t, |z|, |w|, |t| < 1\} \subset \mathbb{C}^3$$

and the projection $\Pi : V \to \Delta$ given by

$$\Pi(z, w, t) = t$$

where Δ is the unit disk.

If $t \in \Delta$ with $t \neq 0$, then the fiber $\Pi^{-1}(t) \subset V$ is an annulus (collar).

If $t = 0$, then the fiber $\Pi^{-1}(t) \subset V$ is two transverse disks $|z| < 1$ and $|w| < 1$.

This is the local model of degeneration of Riemann surfaces.

• *Difficulties*

(1) Find the harmonic Beltrami differentials A_i.
(2) Find the KE metric on the collars.
(3) Estimate the Green function of $(\Box + 1)^{-1}$.
(4) Estimate the norms and error terms.

• *Solution*

We construct approximation solutions on the local model, single out the leading terms and then carefully estimate the error terms one by one.

We have the following precise asymptotic results:

• *Asymptotics in pinching coordinates.*

Theorem. *Let* $(t_1, \cdots t_m, s_{m+1}, \cdots s_n)$ *be the pinching coordinates. Then WP metric* h *has the behaviors:*
(1) $h_{i\bar{i}} = \frac{1}{2}\frac{u_i^3}{|t_i|^2}(1+O(u_0))$ *for* $1 \le i \le m$*;*
(2) $h_{i\bar{j}} = O(\frac{u_i^3 u_j^3}{|t_i t_j|})$ *if* $1 \le i, j \le m$ *and* $i \ne j$*;*
(3) $h_{i\bar{j}} = O(1)$ *if* $m+1 \le i, j \le n$*;*
(4) $h_{i\bar{j}} = O(\frac{u_i^3}{|t_i|})$ *if* $i \le m < j$*.*
Here $u_i = \frac{l_i}{2\pi}$, $l_i \approx -\frac{2\pi^2}{\log|t_i|}$ *and* $u_0 = \sum u_i + \sum |s_j|$*.*

Theorem. *The Ricci metric* τ *has the behaviors:*
(1) $\tau_{i\bar{i}} = \frac{3}{4\pi^2}\frac{u_i^2}{|t_i|^2}(1+O(u_0))$ *if* $i \le m$*;*
(2) $\tau_{i\bar{j}} = O\left(\frac{u_i^2 u_j^2}{|t_i t_j|}(u_i + u_j)\right)$ *if* $i, j \le m$ *and* $i \ne j$*;*
(3) $\tau_{i\bar{j}} = O\left(\frac{u_i^2}{|t_i|}\right)$ *if* $i \le m < j$*;*
(4) $\tau_{i\bar{j}} = O(1)$ *if* $i, j \ge m+1$*.*

Finally we derive the curvature asymptotics:

Theorem. *The holomorphic sectional curvature of the Ricci metric τ satisfies*

$$\widetilde{R}_{i\bar{i}i\bar{i}} = \frac{3u_i^4}{8\pi^4|t_i|^4}(1 + O(u_0)) > 0$$

if $i \leq m$ and

$$\widetilde{R}_{i\bar{i}i\bar{i}} = O(1)$$

if $i \geq m + 1$.

It is important to have the precise estimates of the boundary behaviors and the bounds in the non-degeneration directions.

Next we go to the perturbed Ricci metric

$$\omega_{\tilde{\tau}} = \omega_\tau + C\,\omega_{WP}.$$

we need to compute its curvature:

The curvature formula of the perturbed Ricci metric:

$$
\begin{aligned}
\widetilde{R}_{i\bar{j}k\bar{l}} =& h^{\alpha\bar{\beta}}\left\{\sigma_1\sigma_2\int_{X_s} T(\xi_k(e_{i\bar{j}}))\bar{\xi}_l(e_{\alpha\bar{\beta}})\ dv\right\} \\
& + h^{\alpha\bar{\beta}}\left\{\sigma_1\sigma_2\int_{X_s} T(\xi_k(e_{i\bar{j}}))\bar{\xi}_\beta(e_{\alpha\bar{l}})\ dv\right\} \\
& + h^{\alpha\bar{\beta}}\left\{\sigma_1\int_{X_s} Q_{k\bar{l}}(e_{i\bar{j}})e_{\alpha\bar{\beta}}\ dv\right\} \\
& - \tilde{\tau}^{p\bar{q}}h^{\alpha\bar{\beta}}h^{\gamma\bar{\delta}}\left\{\sigma_1\int_{X_s}\xi_k(e_{i\bar{q}})e_{\alpha\bar{\beta}}\ dv\right\}\times \\
& \left\{\widetilde{\sigma}_1\int_{X_s}\bar{\xi}_l(e_{p\bar{j}})e_{\gamma\bar{\delta}})\ dv\right\} \\
& + \tau_{p\bar{j}}h^{p\bar{q}}R_{i\bar{q}k\bar{l}} + C\,R_{i\bar{j}k\bar{l}}.
\end{aligned}
$$

We noticed that the holomorphic sectional curvature of the perturbed Ricci metric

$$\omega_{\tilde{\tau}} = \omega_\tau + C\,\omega_{WP}$$

remains negative in the degeneration directions when C varies and is dominated by the curvature of the Ricci metric. When C large, the holomorphic sectional curvature of $\tilde{\tau}$ can be made negative in the interior and in the non-degeneration directions near boundary from the curvature of the WP metric. Many detailed analysis involved.

The estimates of the Ricci curvature are long and complicated computations.

The proof of the stability needs the detailed understanding of the boundary behaviors of the KE metric to control the convergence of the integrals of the degrees. Also needed is a basic non-splitting property of the mapping class group.

Bounded geometry: Ricci flow and the higher order estimates of curvature. Injectivity radius bounded.

Remark: (added November 10, 2004)

(1) The most difficult parts of our work are the understanding of the Kähler-Einstein metric, the Ricci and the perturbed Ricci metric, their boundary behaviors, and bounded geometry. These metrics and their behaviors are useful and do have geometric consequences. The equivalences of the classical metrics are easy consequences and a small part of our results.

(2) Yeung recently announced that he could prove the equivalences of the metrics: $\|\cdot\|_C \sim \|\cdot\|_K \sim \|\cdot\|_B \sim \|\cdot\|_T \sim \omega_M$ and ω_{KE} by using **bounded pluri-subharmonic functions** on the Teichmüller space. But in his preprint we received on November 8 he seems to use the similar method to ours in comparing the above metrics by using Yau's Schwarz lemma

and basic definitions of the Bergman, Kobayashi and Carathéodory metric.

This argument, as well as the result, is actually a small and the easiest part of our works. It should be interesting to see how bounded psh function can be used to derive the results.