Towards A Mirror Principle For Higher Genus

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1. Introduction:

Mirror principle is a general method developed in [LLY1]-[LLY4] to compute characteristic classes and characteristic numbers on moduli spaces of stable maps in terms of hypergeometric type series. The counting of the numbers of curves in Calabi-Yau manifolds from mirror symmetry corresponds to the computation of Euler numbers. This principle computes quite general Hirzebruch multiplicative classes such as the total Chern classes.

In the history of mathematics, whenever a function theory or representation theory was well developed, it would come into geometry in a very elegant and substantial way. It is very interesting to see how special functions enter into geometry. Here we only list a few examples:

(a). Symmetric functions $\Leftrightarrow$ characteristic classes.

(b). Trigonometric functions $\Leftrightarrow$ index formulas.

(c). Theta-functions $\Leftrightarrow$ elliptic genus.

(d). Hypergeometric series $\Leftrightarrow$ characteristic numbers on stable map moduli.

(e). q-hypergeometric series $\Leftrightarrow$ K-theory characteristic numbers on stable map moduli.

We like to point out that, in the above, (a), (b) and (c) are representation-theoretic; (d) actually came from the Atiyah-Bott localization formula, equivariant Euler classes, and the geometry of moduli spaces of stable maps. This later connection was made clear in [LLY1]-[LLY4]. (e) should appear in the K-theory computation on stable map moduli space as shown in [LLY2].

Recall that a balloon manifold X is a projective manifold with torus action and isolated fixed points. Let us denote by

$$H = (H_1, \cdots, H_k)$$

a basis of equivariant Kahler classes. Then X is called a balloon manifold if

(1). The restriction $H(p) \neq H(q)$ for any two fixed points $p,\, q \in X$.

(2). The tangent bundle T_pX has linearly independent weights for any fixed point $p \in X$.

The 1-dimensional orbits in X joining every two fixed points in X are called **balloons** which are copies of $\mathbf{P}^1$.

The key ingredients for the proof of the mirror principle are the following three. Their definitions will be given later.

(1). Linear and non-linear sigma model;

(2). Euler data;

(3). Balloons and hypergeometric Euler data.

Note that, as explained in [LLY4], the above ingredients are independent of the genus of the curves, except the hypergeometric Euler data, which for $g > 0$ is more difficult to find out, while for the genus 0 case it can be easily read out from localization at the smooth fixed points of the moduli spaces.

First let us recall the general set-up for mirror principle. Let X be a projective manifold. We introduce the definitions of the main objects needed for the discussions.

Non-linear sigma model, which we denote by $M_d^g(X)$, is the moduli space of stable maps of degree $(1, d)$ and genus g into $\mathbf{P}^1 \times X$:

$$M_d^g(X) = \{(f, C) : \, f : C \to \mathbf{P}^1 \times X\}$$

with C a genus g (nodal) curve and $f(C) \in H^2(\mathbf{P}^1 \times X, \mathbf{Z})$ has bi-degree $(1, d)$, modulo the obvious equivalence. For convenience the degree d will also be used as integers by choosing a basis in $H_2(X,\, \mathbf{Z})$.

Linear sigma model, which we denote by W_d for a toric manifold X was first introduced by Witten and later by Aspinwall-Morrison for computations. It is a large toric manifold. We refer to [LLY3] for a precise definition. Here we only give a simple example.

Example: Let $X = \mathbf{P}^n$, with homogeneous coordinate $[z_0, \cdots, z_n]$. Then the linear sigma model is given by the polynomial space W_d with projective coordinate

$$[f_0(w_0, w_1), \cdots, f_n(w_0, w_1)]$$

where $f_j(w_0, w_1)$ are homogeneous polynomials of degree d.

In the genus 0 case, W_d can be viewed as the simplest compactification of the spaces of degree d maps from $\mathbf{P}^1$ to X. The following basic lemma connects this compactification with a stable map moduli space. A proof can be found in [LLY4].

Lemma: *There exists an explicit equivariant map*

$$\varphi: \ M_d^g(\mathbf{P}^n) \longrightarrow W_d.$$

Here the equivariance is with respect to the induced actions from the torus actions on X and $\mathbf{P}^1$.

Roughly speaking the computation should be on $M_d^g(X)$. But in general $M_d^g(X)$ is very "singular" and complicated. But W_d is smooth and simple, our main strategy is to push-forward everything to W_d through the map φ! The functorial localization formula below is one of the key tricks we used.

Let $M_{g,k}(d,X)$ be the moduli space of stable maps of genus g and degree d with k marked points into X. That is

$$M_{g,k}(d,X) = \{(f, C; x_1, \cdots, x_k): \ f: \ C \to X\}$$

with $x_1, \cdots, x_k$, k points on the genus g (nodal) curve C.

This moduli space may have higher dimension than expected, even worse, its different components may have different dimensions. To compute integrals on such space, we need to first define the integral. For this purpose, we have the notion of virtual cycles: the virtual fundamental class which is first given by Li-Tian [LT] and later by Behrend-Fantechi [BF]. Let us denote by

$$LT_d^g(X) \in A_*(M_d^g)_T,$$

the equivariant analogue of the virtual fundamental cycle which is a class in the equivariant Chow group of cycles of $M_d^g(X)$. Another virtual cycle will also be used:

$$LT_{g,k}(d,X) \in A_*(M_{g,k}(d,X))_T.$$

Now let us introduce the starting data of the argument. We let $V \longrightarrow X$ be an equivariant concavex bundle. The notion of concave bundles was introduced in [LLY1], it represents a direct sum of a positive and a negative bundle on X. From a concavex bundle V, we can induce vector bundles V_d^g on $M_{g,k}(d,X)$ by taking either $H^0(C, f^*V)$ or $H^1(C, f^*V)$, or their direct sum. Let b be a multiplicative characteristic class.

Problem: The main problem of mirror principle is to compute the integral

$$K_d^g = \int_{LT_{g,k}(d,X)} b(V_d^g).$$

More precisely, let λ, q be two formal variables. We would like to compute the generating series,

$$F(q, \lambda) = \sum_{d,g} K_d^g \, \lambda^g q^d$$

in terms of certain natural explicit hypergeometric series. So far we have rather complete **success** for the case of balloon manifolds and genus $g = 0$.

2. Rational curves

The mirror principle for the genus 0 case has been more or less fully developed, which implies almost all of the genus 0 conjectural formulas from string theory [LLY1]-[LLY3]. The most famous corollary is possibly the Candelas formula [Cd]. In this note we will briefly review our approach to the higher genus mirror principle, which is still under progress with partial successes as discussed in [LLY4]. Roughly speaking we now have the following general theorem:

Theorem: *Assume $g = 0$. Mirror principle holds for balloon manifolds and any concavex bundles.*

Remarks: 1. For toric manifolds, the above mirror principle implies almost all mirror conjectural formulas derived from string theory.

2. In the above statement of mirror principle, we need to require *splitting type* on V when restricted onto each balloon $= \mathbf{P}^1$, and certain condition on the first Chern class $c_1(V)$. We refer the reader to [LLY1], [LLY2] for details.

There are many non-split bundle V with given splitting type, such as $T\mathbf{P}^n$; and many equivariant bundles over toric manifolds [LY]. Such bundles will give non-complete intersection Calabi-Yau manifolds, such as Pfaffian variety; moduli space of rank 2 bundles over Riemann surface.

3. The special case of $\mathbf{P}^n$, with V the sum of positive line bundles, b the Euler class, a second approach can be found in [BDPP], [P] following [G]; for V direct sum of positive and negative *line* bundles, see also [E]. In [Ga] a mirror formula was proved by using relative stable maps.

Recently in [Lee], the functorial localization formula [LLY1] and deformations of normal cone were used to derived a mirror formula with V the sum of positive or negative line bundles, and b the Euler class. However, it requires strong restrictions on the first Chern class of V, and it yields no information when V is the trivial bundle.

One of the most interesting corollary of the mirror principle is when we take V to be sum of negative bundles. This gives the so-called **local mirror symmetry**, which is called geometric engineering in [KKV]. The examples include:

(a). Take X to be the de Pezzo surface, $\mathbf{P}^1 \times \mathbf{P}^1$ or $\mathbf{P}^2$. Take $V = K_X$, the canonical line bundle and b the Euler class. In this case, the corresponding hypergeometric series are periods of elliptic curves, which are called the Seiberg-Witten curves [KKV]. Indeed the total space of K_X is an open CY, its mirror is the elliptic curve, the Seiberg-Witten curve.

(b). The simplest but very interesting example is when $X = \mathbf{P}^1$, $V = O(-1) \oplus O(-1)$ and b the Euler class. In this case we have the multiple cover formula of Candelas et al: $K_d = d^{-3}$. When $X = \mathbf{P}^1$, $V = O(-2)$ and b the total Chern class, we get a similar multiple cover formula $K_d = d^{-3}$. Another very interesting example is when $X = \mathbf{P}^2$, $V = O(-3)$ and b the total Chern class [LLY1].

2. Higher genus

In the following we will review our approach to the general mirror principle for higher genus. As one may notice that, almost all of the techniques for genus 0 case work well for higher genus, except the last step of finding the hypergeometric type series which is more difficult in higher genus due to the complicated fixed point moduli spaces.

One of the simple but key techniques used in our approach is the following important **Functorial localization formula**. Let X and Y be two manifolds with torus action.

Lemma: *Let $f: X \to Y$ be an equivariant map. Let $F \subset Y$ be a fixed component, and $E \subset f^{-1}(F)$ be the fixed components in X. Let $f_0 = f|_E$, then for an equivariant cohomology class $\omega \in H_T^*(X)$, we have the identity on F:*

$$f_{0*}\left[\frac{i_E^*\omega}{e_T(E/X)}\right] = \frac{i_F^*(f_*\omega)}{e_T(F/Y)}.$$

It is interesting to note that this functorial localization formula is very much in the spirit of the Riemann-Roch formula. Functorial localization is one of the key ideas in [LLY1] and [LLY2]. This same idea was later used in [B] and [Lee]. We will apply this formula to φ, the collapsing map. Before that let us first work out the fixed points in the nonlinear and linear sigma models with respect to the induced S^1-action from $\mathbf{P}^1$, as well as some of its key properties.

The fixed points in $M_d^g(X)$ induced by the S^1-action on $\mathbf{P}^1$ are given by the components:

$$F_r^{g_1,g_2} = M_{g_1,1}(r,X) \times_X M_{g_2,1}(d-r,X)$$

with $g_1 + g_2 = g$ and $r = 0, \cdots, d$. By considering the pull-back of $b_T(V_d^g)$ through the projection:

$$\pi : M_d^g(X) \to M_{g,0}(d,X)$$

and its restriction to $F_r^{g_1,g_2}$, we have the important

Gluing identity:

$$b_T(V)b_T(V_d^g) = b_T(V_r^{g_1})b_T(V_{d-r}^{g_2}).$$

The collapsing map φ, when restricted to $F_r^{g_1,g_2}$ is just the evaluation map ev into X at the gluing point. The next step is to get the so-called **Euler data** from the above gluing identity. Let us write

$$A_d^g = ev_*[\frac{i^*\pi^*b_T(V_d^g) \cap LT_{g,1}(d,X)}{e_T(F_d^{g,0}/M_d^g(X))^v}]$$

which comes from the left hand side of the *Functorial Localization Formula.* Here we have actually used a virtual version of the functorial localization formula, which is proved by using the virtual Atiyah-Bott formula as generalized in [GP]. The denominator $e_T(F_d^{g,0}/M_d^g(X))^v$ denotes the virtual equivariant Euler class. See [LLY3] and [LLY4].

Let us form the generating series:

$$A_d = \sum_g A_d^g\,\lambda^g, \quad A = \sum_d A_d\,e^{dt}$$

¿From gluing identity and the functorial localization formula we can derive the following identity:

$$b_T(V)\ \cdot i_r^* A_d^{g_1,g_2} = \bar{A}_r^{g_1}\ \cdot A_{d-r}^{g_2} \quad (*)$$

where $i_r^* A_d^{g_1,g_2}$ is the local term from the localization of $b_T(V_d^g)$ onto $F_r^{g_1,g_2}$, and $\bar{A}_r$ denotes the switch of sign: $\alpha \to -\alpha$. Here α is the weight of the S^1-action induced from the action on $\mathbf{P}^1$. This then gives us quadratic relations among the A_d's. See [LLY4] for the details.

The right hand side of the functorial localization formula is the localization of the push-forward class by φ:

$$\varphi_*[b_T(V_d^g) \cap LT_d^g(X)] \in A_*(W_d)_T \quad (**)$$

which is a polynomial class in α. Note that A_d^g is actually a rational class in α. Through functorial localization formula and localization on W_d, we derive, from the gluing identity, that $\{A_d\}$ *is an Euler data.*

Here **Euler data**, roughly speaking, are the sequences of classes like A_d with properties like (*) and (**). The connection between (*) and (**) is the functorial localization formula. From the above discussion, we see that any triple (X, V, b) induces an Euler data through the functorial localization formula.

On the other hand, we know that knowing A_d^g is equivalent to knowing K_d^g, as given by the following:

Lemma: *We have the formula:*

$$\alpha^{g-3}(2-2g-d\cdot t)K_d^g = \int_X e^{-t\cdot H/\alpha}A_d^g.$$

So the problem is reduced to the computation of the Euler data A_d. The next step in our approach is to approximate A_d by restricting to "smooth part" or "generic part" of $M_{g,k}(d,X)$.

When the genus $g=0$, by localization to smooth fixed points, the multiple covers of the balloons, which are those complex 1-dimensional orbits in X. When restricting the A_d to those smooth fixed points in $M_{0,1}(d,X)$, we get another class B_d which is an explicit hypergeometric type cohomology class. Here we just illustrate by a typical example:

Example: Let $X=\mathbf{P}^n$, $V=O(l)$ and b=Euler class. Then we have

$$B_d = \frac{\prod_{m=0}^{ld}(lH-m\alpha)}{\prod_{m=1}^{d}(H-m\alpha)^{n+1}}.$$

The general toric case is very similar, and B_d is also read out from localization on the balloons. Here for general vector bundle V, the splitting type comes in. See [LLY3] for further details.

By applying localization formula on the linear sigma model W_d of X, we find that B_d is also an Euler data. And we know that $A_d=B_d$ at the smooth points, which we called them linked. Together with an Lagrange interpolation type argument, we derive the following uniqueness lemma by using localization again:

Lemma: *If* $\deg_\alpha(A_d-B_d)\le -2$*, then* $A_d=B_d$.

That is to say that B_d determines A_d up to degree -2. But in general B_d has higher degree, instead this degree is zero. Then we can always find a so-called mirror transformation to decrease its degree to -2. Here is one of the typical example of the mirror formula as a corollary of mirror principle:

Example: Let X be a toric manifold; and consider the case of $g=0$. Let $D_1,..,D_N$ be the T-invariant divisors, and V the direct sum of positive line bundles: $V=\oplus_i L_i$, $c_1(L_i)\ge 0$ and $c_1(X)=c_1(V)$.

Let $b(V)=e(V)$, $\Phi(T)=\sum K_d\, e^{d\cdot T}$, and

$$B(t) = e^{-H\cdot t}\sum_d \prod_i \prod_{k=0}^{\langle c_1(L_i),d\rangle} (c_1(L_i)-k)$$

$$\times \frac{\prod_{\langle D_a,d\rangle<0}\prod_{k=0}^{-\langle D_a,d\rangle-1}(D_a+k)}{\prod_{\langle D_a,d\rangle\geq 0}\prod_{k=1}^{\langle D_a,d\rangle}(D_a-k)}\ e^{d\cdot t}.$$

Then the mirror principle implies that there are explicitly computable functions $f(t)$, $g(t)$, such that

$$\int_X \left(e^f B(t) - e^{-H\cdot T}e(V)\right) = 2\Phi - \sum T_i \frac{\partial \Phi}{\partial T_i}$$

where $T = t + g(t)$. From this formula we can determine Φ uniquely.

The whole argument is actually genus independent except finding B_d. The problem is that for $g > 0$, the good fixed points are given by the Deligne-Mumford moduli space of stable curves $\mathcal{M}_{g,1}$. And when localizing A_d to such fixed points, we get explicit Hodge integrals on $\mathcal{M}_{g,1}$, which are all explicitly computable. Our approach works well until the *last step*: we can not figure out a simple B_d from the integral on $\mathcal{M}_{g,1}$, which is again an Euler data, to approximate A_d.

But the fact that A_d is an Euler data already puts very strong restriction on such sequences, and this determines it up to certain degree. Such restrictions are all quadratic and compatible with mirror symmetry for higher genus by Vafa et al. Even if we take X a single point in our non-linear sigma model, we already get strong information on the Hodge integrals on $\mathcal{M}_{g,1}$. This was worked out in [FP]. More interesting results are obtained in [TZ]. At this point we are trying to design more refined localization involving $\mathcal{M}_g$ to find some more refined relations among the A_d's.

Euler data is a very general notion, it can include general Gromov-Witten invariants. We can consider marked points to the moduli spaces and add the pull-back classes to the A_d's. More precisely we can try to compute integrals of the form:

$$K_{d,k}^g = \int_{LT_{g,k}(d,X)} \prod_j ev_j^*\omega_j \ \cdot b(V_d^g)$$

where $\omega_j \in H^*(X)$.

By introducing the generating series with summation over k, we can still get Euler data. The *Ultimate Mirror Principle* we are searching for is the following statement: *Compute this series by explicit hypergeometric series!* Our discussion above has reduced this to the problem of finding the hypergeometric Euler data B_d's.

4. Concluding remarks:

Now we like to discusss some related problems to our above discussions. we will not give details here.

(1). *Counting holomorphic discs:* The boundary of these holomorphic discs lie in certain special Lagrangian sub-manifold, which is some vanishing cycle, in X. We hope to extend mirror principle to deal with such problems. Nonlinear sigma model has been studied by Fukaya et al, and linear sigma model has been worked out in string theory. In this situation both sigma models have boundaries. The string theorists Vafa et al have made several interesting conjectures. Some progresses have been made in [KL] and [LS].

(2). *The Gopakumar-Vafa formula:* This formula [GV] reinterpretes, via physics, the rational number K_d^g in terms of certain integer valued instanton numbers n_d^g, generalizing the multiple cover formula for rational curves. In particular, in the genus zero case, this gives rise to integer series expansions for Yukawa couplings. The question of integrality and divisibility of these series expansions were first studied in [LY], as a special of Lian-Yau's integrality conjecture. Using the formula of [GV] and this conjecture as a guide, we hope to construct hypergeometric Euler data, which is linked to A_d.

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