(1) Consider the system $\dot{x} = \sin(x) + \cos(x)$.

(a) Sketch the corresponding vector field on the line, find all fixed points of the system, and classify the stability of these fixed points.

(b) Show that $\dot{x}$ gives a well-defined vector field on the circle and sketch the corresponding vector field on the circle.

(c) Find the acceleration $\ddot{x}$ of the flow $x(t)$ as a function of $x$ and find the points on the circle where the flow has maximum positive acceleration.

Solution:

(a) The stable fixed points are of the form $x^* = \frac{3\pi}{4} + 2\pi k$ and the unstable fixed points are of the form $x^* = \frac{7\pi}{4} + 2\pi k$, for $k$ an integer.

(b) For any integer $k$, we have

$$\sin(x + 2\pi k) + \cos(x + 2\pi k) = \sin(x) + \cos(x)$$

Thus, indeed, the vector field associated to $\dot{x}$ is well-defined on the circle.

(c) Applying the chain rule, the acceleration is given by

$$\ddot{x} = \frac{d}{dt}(\dot{x})$$

$$= \frac{d}{dx}(\sin(x) + \cos(x)) \cdot \frac{dx}{dt}$$

$$= (\cos(x) - \sin(x))(\sin(x) + \cos(x))$$

$$= \cos^2(x) - \sin^2(x)$$

So, the acceleration is maximal when $\cos^2(x) = 1$ and $\sin^2(x) = 0$, namely when $x = 0$ or $x = \pi$.

(2) Show that the system $\dot{x} = x^{\frac{5}{6}}$ with $x(0) = 0$ has infinitely many solutions $x(t)$.

Hint: First find two distinct solutions and then “glue” these together at arbitrary $t_0$

Solution:

It is clear the $x(t) = 0$ is one solution of the system. However, we may also solve the system analytically by separating variables and integrating. Using this method, the system yields $x^{-\frac{1}{6}}dx = dt$ and so $6x^{\frac{1}{6}} = t + C$. Since $x(0) = 0$, the constant $C$ is zero, and thus, solving for $x$ we have $x(t) = (\frac{1}{4}t)^6$ as a second solution to the system.

Now choose an arbitrary positive real number $t_0$ and consider the function

$$x(t) = \begin{cases} 
(\frac{1}{4}(t - t_0))^6 & \text{if } t \geq t_0 \\
0 & \text{if } -t_0 < t < t_0 \\
(\frac{1}{4}(t + t_0))^6 & \text{if } t \leq t_0 
\end{cases}$$
Then, we have \( x(0) = 0 \) and

\[
\dot{x} = \begin{cases} 
\left( \frac{1}{6} (t - t_0) \right)^5 & \text{if } t \geq t_0 \\
0 & \text{if } -t_0 < t < t_0 \\
\left( \frac{1}{6} (t + t_0) \right)^5 & \text{if } t \leq t_0 
\end{cases}
\]

So the proposed \( x(t) \) is indeed a solution of the system \( \dot{x} = x^{\frac{5}{6}} \). Since \( t_0 \) can be chosen to be any positive real number, we have, in this way, produced infinitely many solutions of the given system.

(3) Prove that periodic solutions for a vector field on a line are impossible using the following analytic method:

Suppose \( x(t) \) is a nontrivial periodic solution to a system \( \dot{x} = f(x) \), namely \( x(t) = x(t+C) \) for some minimal \( C > 0 \). Consider the integral \( \int_{t}^{t+C} f(x) \frac{dx}{dt} dt \).

(a) Simplify the integral using the chain rule.

(b) Simplify the integral by plugging in for \( f(x) \) and \( \frac{dx}{dt} \).

Conclude that such a solution is impossible.

**Solution:**

(a) By the chain rule, \( \int_{t}^{t+C} f(x) \frac{dx}{dt} dt = \int_{x(t)}^{x(t+C)} f(x) dx \), which equals zero because \( x(t + C) = x(t) \).

(b) By plugging in \( \dot{x}(t) \) for the \( f(x) \) and \( \frac{dx}{dt} \) terms, we get \( \int_{t}^{t+C} f(x) \frac{dx}{dt} dt = \int_{t}^{t+C} \dot{x}^2 dt \), which must be strictly greater than zero, since by assuming \( x(t) \) is nontrivial periodic we ensure \( \dot{x} \neq 0 \).

But this is impossible. We cannot have \( \int_{t}^{t+C} f(x) \frac{dx}{dt} dt \) both equal to zero and strictly greater than zero. Therefore such a periodic solution to the system \( \dot{x} = f(x) \) cannot exist.

(4) Consider the system \( \dot{x} = ax - x^3 \) where \( a \) may be positive, negative, or zero. Find the fixed points of \( \dot{x} \) and classify their stability using the following methods:

(a) Graphing the potential \( V(x) \) and determining its equilibrium points

(b) Using linear stability analysis or, if linear stability analysis fails, graphical arguments

**Solution:**

(a) We need to solve \(-\frac{dv}{dx} = ax - x^3 \). Integrating and multiplying by -1 yields

\[
V(x) = \frac{1}{4} x^4 - \frac{1}{2} ax^2
= \frac{1}{4} x^2 (x^2 - 2ax)
\]

When \( a < 0 \), we have a single equilibrium located at \( x = 0 \) and it is stable.

When \( a = 0 \), we also have a single equilibrium point at \( x = 0 \) and it is stable.

When \( a > 0 \) we have three equilibrium points: stable ones at \( x = \pm \sqrt{a} \) and an unstable one at \( x = 0 \).

(b) Write \( \dot{x} = f(x) \), as usual. Then \( f(x) = ax - x^3 = x(a - x^2) \) and \( f'(x) = a - 3x^2 \).

If \( a < 0 \), we have a single fixed point \( x^* = 0 \) and since \( f'(0) = a \) it is stable.
If \( a = 0 \), we have also a single fixed point \( x^* = 0 \), but now since \( f'(0) = a = 0 \), we cannot use linear stability analysis to determine its stability. However, since in this case \( f(x) = -x^3 \) it is easy to see graphically that the fixed point is stable.

If \( a > 0 \) we have three fixed points: \( x^* = \pm \sqrt[3]{a} \) and \( x^* = 0 \). Since \( f'(\pm \sqrt[3]{a}) = -2a = -2a \) and \( f'(0) = a \), the points \( x^* = \pm \sqrt[3]{a} \) are stable and the point \( x^* = 0 \) is unstable.

(5) Consider the system \( \dot{x} = rx - \frac{x}{1+x^2} \). Find the values of \( r \) at which bifurcations occur and sketch the bifurcation diagram of the fixed points \( x^* \) of the system versus \( r \).

**Solution:**

We wish to determine changes in the number and stability of the fixed points of \( \dot{x} \) as we vary \( r \). Namely we want to consider for how many \( x \) we have \( 0 = rx - \frac{x}{1+x^2} \). Multiplying both sides of this equation by \( 1 + x^2 \) and simplifying yields \( 0 = rx(x^2 + \frac{r-1}{r}) \).

Then when \( r \leq 0 \), the only solution to this equation is at \( x = 0 \). When \( 0 < r < 1 \), there are three solutions: \( x = \pm \sqrt{\frac{r-1}{r}} \) and \( x = 0 \). When \( r > 1 \), we again have that the only solution is \( x = 0 \).

Translating this into the language of fixed points, when \( r \leq 0 \), the only fixed point of \( \dot{x} \) is at \( x^* = 0 \) and by graphing we see it is stable. When \( 0 < r < 1 \), there are three fixed points: \( x^* = \pm \sqrt{\frac{r-1}{r}} \), which are unstable, and \( x^* = 0 \) which is stable. When \( r > 1 \), the only fixed point is at \( x^* = 0 \) and it is unstable.

So there are bifurcations at \( r = 0 \) and at \( r = 1 \). The one at \( r = 1 \) is a subcritical pitchfork bifurcation and the one at \( r = 0 \) we don’t have a name for.

(6) Consider the system \( \dot{x} = h + rx - x^2 \).

(a) Sketch the bifurcation diagram for the system \( \dot{x} \) when \( h \) is positive, negative, and zero.

(b) Determine the regions of the \((r, h)\) plane corresponding to qualitatively different vector fields and identify the bifurcations occurring on the boundaries of these regions.

**Solutions:**

(a) In order to sketch the bifurcation diagrams, we need to understand the fixed points of the system, namely the solutions to \( 0 = h + rx^* - x^*^2 \). Applying the quadratic formula yields \( x^* = \frac{r \pm \sqrt{r^2 + 4h}}{2} \). We then graph this for the various \( h \) considered.

Note, for instance, that when \( h = 0 \) our system is \( \dot{x} = rx - x^2 \), which is the normal form for a transcritical bifurcation, thus we already know what the bifurcation diagram in this case looks like.

(b) We have the explicit formula \( x^* = \frac{r \pm \sqrt{r^2 + 4h}}{2} \) for the fixed points of \( \dot{x} \) in terms of \( r \). So, in this case, the number of fixed points of the system is completely determined by the \((r^2 + 4h)\)-term in this expression. Namely, when \( r^2 + 4h > 0 \) there are two distinct fixed points, given by \( x^* = \frac{r - \sqrt{r^2 + 4h}}{2} \) and \( x^* = \frac{r + \sqrt{r^2 + 4h}}{2} \). When \( r^2 + 4h = 0 \), there is a single fixed point \( x^* = \frac{r}{2} \). When \( r^2 + 4h < 0 \), there are no fixed points.

A change in stability occurs, as addressed in part (a) when \( h = 0 \) and \( r = 0 \) because we have a transcritical bifurcation there.

Namely, in the \((r, h)\)-plane you should draw the curve \( h = -\frac{r^2}{4} \) and the point \((0, 0)\). Underneath, the curve \( h = -\frac{r^2}{4} \) there are no fixed points, on the curve there is one fixed point, and above the curve there are 2 fixed points. When \( h = 0 \), the two fixed points occurring change their stability at \( r = 0 \). Namely, saddle-point bifurcations...
occur along the curve $h = -\frac{r^2}{4}$ and a transcritical bifurcation happens at the point $(0, 0)$. 