MATH 134: PRACTICE FINAL SOLUTIONS

(1) Consider the system \( \dot{x} = 2 \sin(x) - \sqrt{2} \).

(a) Plot the potential function \( V(x) \) of the system and identify the equilibrium points and their stability.

(b) Confirm your answer using linear stability analysis on the fixed points of the system.

(c) Plot the vector field corresponding to the given system and then plot the possible solutions to the system for various initial conditions.

Solution:

(a) The potential function \( V(x) \) is \( 2 \cos(x) + \sqrt{2}x \). The unstable equilibrium points occur at \( x = \frac{\pi}{4} + 2\pi k \) and the stable equilibrium points occur at \( x = \frac{3\pi}{4} + 2\pi k \) for all integers \( k \).

(b) Letting \( f(x) = 2 \sin(x) - \sqrt{2} \) we have \( f'(x) = 2 \cos(x) \), and thus \( f'(\frac{\pi}{4} + 2\pi k) = \sqrt{2} > 0 \) and \( f'(\frac{3\pi}{4} + 2\pi k) = -\sqrt{2} < 0 \) for any integer \( k \). Thus, indeed, \( x^* = \frac{\pi}{4} + 2\pi k \) (for \( k \in \mathbb{Z} \)) are the unstable fixed points of the system and \( x^* = \frac{3\pi}{4} + 2\pi k \) are the stable fixed points of the system.

(2) Consider the system \( \dot{x} = rx - \cos(x) \).

(a) When \( r = 0 \), sketch the corresponding vector field on the line, find all fixed points of the system, and classify the stability of these fixed points.

(b) Find the values of \( r \) for \( r > 0 \) at which bifurcations occur and classify the occurring bifurcations.

(c) Find the values of \( r \) for \( r < 0 \) at which bifurcations occur and classify the occurring bifurcations.

(d) Plot the bifurcation diagram for the system, making sure to indicate stability of the various branches of fixed points.

Solution:

(a) When \( r = 0 \) the unstable fixed points of the system occur at \( x^* = \frac{\pi}{2} + 2\pi k \) and the stable fixed points occur at \( x^* = \frac{-\pi}{2} + 2\pi k \) for \( k \in \mathbb{Z} \).

(b) A saddle node bifurcation occurs at \( r = r_c \) when \( 1 = r_c(1-n\pi) \) for \( n \) positive and even and when \( -1 = r_c(1-\pi) \) for \( n \) positive and odd. Namely, we have a saddle node bifurcation at \( r_c = \frac{1}{n\pi} \) for any positive integer \( n \).

(c) A saddle node bifurcation occurs at \( r = r_c \) when \( 1 = r_c(n\pi) \) for \( n \) positive and even and when \( -1 = r_c(n\pi) \) for \( n \) positive and odd. Namely, we have a saddle node bifurcation at \( r_c = \frac{1}{n\pi} \) for any positive integer \( n \).

(3) Find the fixed points, sketch the nullclines, the vector field, and a plausible phase portrait for the system \( \dot{x} = x - x^3, \dot{y} = -y \).

Solution:
The fixed points occur at \((-1,0), (0,0),\) and \((1,0)\) and they are a stable node, a saddle point, and a stable node respectively. The vertical nullclines occur at \(x = -1, x = 0,\) and \(x = 1\) and the horizontal nullcline occurs at \(y = 0.\) This means, in this case, all nullclines are also trajectories. This fact makes sketching the phase portrait fairly straightforward.

(4) Consider the system \(\dot{x} = x^2 - 2x + y, \dot{y} = 3x^2 + y.\)

(a) Find all fixed points of the system and classify them.

(b) Show that the line \(y = 3x\) is invariant.

(c) Show that \(|3x(t) - y(t)| \to 0\) as \(t \to \infty\) for all other trajectories.

\textit{Hint: Consider a differential equation for }3x - y.\textit{ }

(d) Sketch the phase portrait.

\textbf{Solution:}

(a) The system has two fixed points: one at \((0,0)\) and one at \((-1,-3).\) Considering Jacobians reveals that \((0,0)\) is a saddle point and \((-1,-3)\) is a stable node.

(b) Let \((x_0,y_0)\) be a point on the line \(y = 3x.\) Then we must have \(y_0 = 3x_0\) and so the vector at \((x_0,y_0)\) defined by \((\dot{x}, \dot{y})\) is \((x_0^2 + x_0, 3x_0^2 + 3x_0),\) which indeed lies on the line \(y = 3x.\) Hence, the line \(y = 3x\) is invariant.

(c) Let \(u(t) = 3x(t) - y(t).\) Then we have

\[
\dot{u} = 3\dot{x} - \dot{y} \\
= 3x^2 - 6x + 3y - 3x^2 - y \\
= -2(3x - y) \\
= -2u
\]

Solving the differential equation \(\dot{u} = -2u\) yields \(u(t) = u_0 e^{-2t}.\) Since \(u(t) \to 0\) as \(t \to \infty,\) we indeed have \(|3x(t) - y(t)| \to 0\) as \(t \to \infty.\)

(5) Determine if the following systems have a nonlinear center at the origin.

(a) \(\dot{x} = -y - x^2, \dot{y} = x\)

(b) \(\ddot{x} + x + \epsilon x^3 = 0\) where \(\epsilon > 0\)

\textbf{Solution:}

(a) The given system is a reversible system. We can show this by showing, for \(\dot{x} = f(x,y)\) and \(\dot{y} = g(x,y),\) that \(f(-x,y) = f(x,y)\) and \(g(-x,y) = -g(x,y)\) (note that here we are reversing the roles of \(x\) and \(y\) in the usual definition of reversible). This is, indeed, the case when \(f(x,y) = -y - x^2\) and \(g(x,y) = x.\) Since the system is reversible, in order to show that the origin is a nonlinear center it is sufficient to check that the origin is a linear center. This is easily verified since the Jacobian of the system is \(\begin{pmatrix} -2x & -1 \\ 1 & 0 \end{pmatrix},\)

which at the origin is just \(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.\) This matrix has trace 0 and determinant 1, indicating that the origin is, indeed, a linear center. Hence, the origin is, in fact, a nonlinear center.
(b) Let \( y = \dot{x} \) and rewrite the system as \( \dot{x} = y, \dot{y} = -x - \epsilon x^3 \). This yields potential function \( V(x) = \frac{1}{2}x^2 + \frac{\epsilon}{4}x^4 \). Let \( E(x, y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{\epsilon}{4}x^4 \). Then we have

\[
\frac{dE}{dt} = yy + xx + \epsilon x^3 \dot{x} \\
= x\ddot{x} + x\dot{x} + \epsilon x^3 \dot{x} \\
= x(\ddot{x} + x + \epsilon x^3)
\]

So \( E \) is a conserved quantity of the system and thus the system is conservative. Moreover, \( E \) has hessian given by \( \begin{pmatrix} 3\epsilon x^2 + 2 & 0 \\ 0 & 1 \end{pmatrix} \), which at \((0, 0)\) is \( \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \), hence by the second partial derivative test implies \((0, 0)\) is a local minimum of \( E \). Then, since the origin is an isolated fixed point of the system, the conditions of Theorem 6.5.1 are satisfied, and so the origin is a nonlinear center.

(6) Sketch the phase portrait for the system given in polar coordinates by \( \dot{r} = r \cos(r), \dot{\theta} = 1 \).

**Solution:**

The phase portrait should have a fixed point at \( r = 0 \) and closed circular orbits at \( r = \frac{\pi}{2} + \pi k \) for every integer \( k \). When \( k \) is even these circular orbits are stable limit cycles and when \( k \) is odd they are unstable limit cycles.

(7) Show that the system \( \dot{x} = x^2 + 2\cos(y) - 1, \dot{y} = 1 - e^{-y^2} \) has no closed orbits.

**Solution:**

The system has no fixed points. Since any closed orbit contains a fixed point, the system cannot have any closed orbits.

(8) Consider the system \( \dot{x} = x^2 - y - 1, \dot{y} = y(x - 2) \).

(a) Find and classify the fixed points of the system.

(b) Show the system has no closed orbits.

*Hint: Show that the lines through pairs of fixed points are all trajectories.*

(c) Sketch the phase portrait.

**Solution:**

(a) The fixed points of the system are \((1, 0), (-1, 0), \) and \((2, 3)\). The Jacobian of the system is \( \begin{pmatrix} 2x & -1 \\ y & x - 2 \end{pmatrix} \).

At \((1, 0)\), the Jacobian is \( \begin{pmatrix} 2 & -1 \\ 0 & -1 \end{pmatrix} \), which has \( \tau = 1 \) and \( \Delta = -2 \), meaning \((1, 0)\) is a saddle node.

At \((-1, 0)\), the Jacobian is \( \begin{pmatrix} -2 & -1 \\ 0 & -3 \end{pmatrix} \), which has \( \tau = -5 \) and \( \Delta = 6 \), meaning \((-1, 0)\) is a stable node.

At \((2, 3)\), the Jacobian is \( \begin{pmatrix} 4 & -1 \\ 3 & 0 \end{pmatrix} \), which has \( \tau = 4 \) and \( \Delta = 3 \), meaning \((2, 3)\) is an unstable node.
(b) The 3 lines through the pairs of fixed points are \( y = 0, \ y = x + 1, \) and \( y = 3x - 3.\) Note that on \( y = 0 \) we have \((\dot{x}, \dot{y}) = (x^2 - 1, 0),\) on \( y = x + 1 \) we have \((\dot{x}, \dot{y}) = (x^2 - x - 2, x^2 - x - 2),\) and on \( y = 3x - 3 \) we have \((\dot{x}, \dot{y}) = (x^2 - 3x + 2, 3x^2 - 9x + 6).\) Hence all three of these lines are invariant lines, namely they correspond to trajectories. If the system had a closed orbit, it would have to contain a fixed point. But if it contained any of the above fixed points, it would have to cross at least two of the above invariant lines. Since trajectories cannot intersect, this is impossible. Therefore, the system cannot have any closed orbits.

(9) Consider the system \( \dot{x} = y + ax(1 - 2b - r^2), \ \dot{y} = -x + ay(1 - r^2), \) where \( a \) and \( b \) are parameters such that \( 0 \leq a \leq 1 \) and \( 0 \leq b \leq \frac{1}{2} \) and where \( r^2 = x^2 + y^2, \) as usual.

(a) Using that \( r = \sqrt{x^2 + y^2} \) and \( \theta = \tan^{-1}(\frac{y}{x}) \) we get:

\[
\dot{r} = \frac{1}{2} (x^2 + y^2) \frac{1}{r} (2x \dot{x} + 2y \dot{y})
= \frac{1}{r} (x \dot{x} + y \dot{y})
= \frac{1}{r} (xy + ax^2(1 - 2b - r^2) - xy - ary^2(1 - r^2))
= \frac{1}{r} (-2abx^2 + a(x^2 + y^2)(1 - r^2))
= \frac{1}{r} (-2abr^2 \cos^2(\theta) + ar^2(1 - r^2))
= ar(-2b \cos^2(\theta) + (1 - r^2))
\]

\[
\dot{\theta} = \frac{1}{1 + (\frac{y}{x})^2} \cdot \frac{xy - y \dot{x}}{x^2}
= \frac{1}{r^2} (-2ax^2 + axy(1 - r^2) - y^2 - ary(1 - 2b - r^2))
= \frac{1}{r} (-2ax^2 + ay^2 + 2abxy)
= \frac{1}{r^2} (-r^2 + 2abr^2 \cos(\theta) \sin(\theta))
= ab \sin(2\theta) - 1
\]

(b) Note that when \( r > 1 \) we have \( \dot{r} < ar(-2b \cos^2(\theta) + (1 - 1)) = -2abr \cos^2(\theta) < 0 \) and when \( r < \sqrt{1 - 2b} \) we have \( \dot{r} > ar(-2b(1) + 1 - (1 - 2b)) = ar(0) = 0. \) Thus we have a trapping region \( R \) given by \( \sqrt{1 - 2b} \leq r \leq 1, \) meaning any trajectory starting in \( R \) must remain in \( R \) for all time. Moreover, we know that the system has a fixed point at \( r = 0. \) If the system had a fixed point \( (r^*, \theta^*) \) for \( r \neq 0, \) we would need \( \theta = 0 \) at this point. Namely, we would have \( \theta^* = \frac{1}{2} \sin^{-1}(\frac{1}{ab}) \). However, note that \( 0 \leq a \leq 1 \) and \( 0 \leq b \leq \frac{1}{2}, \) namely \( ab \leq \frac{1}{2} \) and so \( \frac{1}{ab} \geq 2. \) But this means \( \sin^{-1}(\frac{1}{ab}) \) does not exist.
Hence the system has no other fixed points besides the one at $r = 0$ and so, certainly, our $R$ contains no fixed points. Therefore, applying the Poincaré-Bendixson Theorem, the system must have a limit cycle contained in $R$.

The period of any limit cycle of the system is given by:

$$T(a, b) = \int dt = \oint dt = \oint \frac{dt}{d\theta} d\theta = \int_{0}^{2\pi} \frac{1}{ab\sin(2\theta) - 1} d\theta$$