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Ornstein–Zernike Behavior for Self-Avoiding Walks at All Noncritical Temperatures

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Abstract. We prove that the self-avoiding walk has Ornstein–Zernike decay and some related properties for all noncritical temperatures at which the model is defined.

1. Introduction

The original derivation of the long-distance behavior of a two-point correlation dates back to the classic work of Ornstein and Zernike [1] in 1914. Although Ornstein and Zernike examined only the classical fluid, it has since been realized that their conclusions should apply also to the two-point functions of many lattice spin systems and Euclidean quantum field theories. Moreover, by means of the Källen–Lehman representation, it has been demonstrated that there is a relatively straightforward relationship between the decay of the two-point function and the particle spectrum of the associated field theory (see, e.g. [13]). Motivated by this connection, there has been much interest in rigorously establishing Ornstein–Zernike decay for a variety of spin systems and lattice field theories [2–23]. Unfortunately, the vast majority of this work has established this decay only in a perturbative regime (e.g., high or low temperature or strong coupling).

In this paper, we consider self-avoiding walks, and prove Ornstein–Zernike decay and some related properties for all noncritical temperatures. Our method relies on the approach initiated in [24] and [2] (see also [9–11, 22, 23]), which shows that the original ideas of Ornstein and Zernike may be implemented whenever one can define a *direct correlation function* with a strictly larger decay rate than that of the two-point function. Here we prove such an assertion by constructing both a direct correlation function and a set of rescaled variables which bound this function. We then show that the rescaled variables obey a (renormalized) *Ornstein–Zernike inequality* which provides a bound on their decay rate and hence on that of the direct

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correlation function. With certain modifications, we expect that such techniques can be applied to other models of random aggregates, and possibly also (using, e.g., the random graph expansions of φ^4 theory [25, 26]) to quantum theories with matter fields.

Our renormalization approach is to be contrasted with the work of [11] in which the decay of the two-point function was established nonperturbatively for a particular class of random surfaces by the introduction of vertex terms for the Ornstein–Zernike equation. It is likely that a nonperturbative proof of Ornstein–Zernike decay for general surface models will require both renormalization arguments along the lines of those presented here and the incorporation of vertex terms as in [11].

2. Definitions, Statement of Results and Strategy of the Proof

In this paper, we will study and characterize the noncritical behavior of various *generating functions* for self-avoiding walks of $\mathbb{Z}^d$. Let $\mathscr{W}$ denote the set of all self-avoiding walks (S.A.W.'s) of $\mathbb{Z}^d$. The generating function for some subset $S \subset \mathscr{W}$ is defined by

$$G_S(\beta) = \sum_{w \in S} e^{-\beta|w|}, \tag{2.1}$$

where $\beta \in [0, \infty]$ is a parameter, and $|w|$ denotes the length (i.e., number of steps) of the walk $w \in S$. Of principal concern is the generating function for walks between two specified points x, $y \in \mathbb{Z}^d$. This is called the *two-point function* and is denoted by

$$G_{xy}(\beta) = \sum_{w: x \to y} e^{-\beta|w|}. \tag{2.2}$$

Here the sum is over all walks from x to y in $\mathscr{W}$, and by convention, $G_{xx}(\beta) = 1$. When one endpoint is the origin of coordinates, we use the special notation $G_x(\beta) \equiv G_{0x}(\beta)$.

It is also of interest to consider the generating function for walks from the origin to some $(d-1)$-dimensional hyperplane. Without loss of generality, we henceforth choose the hyperplane to be perpendicular to the x_1-axis: $\mathbb{P}_L = \{x \in \mathbb{Z}^d | x_1 = L\}$, so that $x \in \mathbb{P}_L$ may be written in the form $x = (L; a)$ with $a = (x_2, \ldots, x_d)$. The corresponding *point-to-plane* generating function is given by

$$\mathbb{G}_L(\beta) = \sum_{x \in \mathbb{P}_L} G_x(\beta) = \sum_{a \in \mathbb{Z}^{d-1}} G_{(L;a)}(\beta). \tag{2.3}$$

Finally, the susceptibility is defined by

$$\chi(\beta) = \sum_{x \in \mathbb{Z}^d} G_x(\beta) = \sum_{L \in \mathbb{Z}} \mathbb{G}_L(\beta), \tag{2.4}$$

which is just the generating function for all walks which start at the origin.

In the above definitions, we have tacitly assumed that β is large enough to guarantee that the above generating functions are finite. It turns out that there is a (dimension-dependent) constant, β_c, such that for $\beta > \beta_c$, the quantities (2.2)–(2.4) are finite, while if $\beta < \beta_c$, these quantities diverge. (Divergence of χ at β_c follows from the results of [27]; that the correlations are infinite below the same point is proved in [23].) Our analysis pertains to the region $\beta > \beta_c$.

Our results concern the long-distance behavior of the generating functions $\mathbb{G}_L$ and $G_{(L;a)}$. First we prove that $\forall \beta > \beta_c$, $\mathbb{G}_L(\beta)$ has "pure exponential decay" in the sense that there exist constants $m(\beta) > 0$, $\Delta(\beta) > 0$ and $C(\beta) > 1$ such that

$$|\mathbb{G}_L(\beta)e^{+m(\beta)L} - C(\beta)| \leqq e^{-\Delta(\beta)L}. \tag{2.5}$$

The decay rate $m(\beta)$ may be identified as the *mass*, and $\Delta(\beta)$ as the *upper gap* of the spectrum.

Next we prove that for $\beta > \beta_c$, there exist constants $\alpha(\beta) > 0$ and $\varepsilon(\beta) > 0$ such that for all $a \in \mathbb{Z}^{d-1}$ satisfying $|a| < \varepsilon(\beta)L$, $G_{(L;a)}(\beta)$ admits an asymptotic expansion in powers of $L^{-1/2}$, the leading term of which is given by

$$G_{(L;a)}(\beta) = C(\beta)[\alpha(\beta)\pi L]^{-(d-1)/2} e^{-m(\beta)L} e^{-a^2/\alpha(\beta)L}[1 + O(L^{-1/2})]. \tag{2.6}$$

Observe that the ratio $G_{(L;a)}/\mathbb{G}_L$ represents the hitting distribution of endpoints in the plane $\mathbb{P}_L$. Thus (2.5) and (2.6) imply that the hitting converges to a normal distribution under the scaling $a \to a[\alpha L]^{-1/2}$. Ornstein–Zernike decay is given by the factor of $L^{-(d-1)/2}$ in (2.6), which is simply the normalization of the Gaussian distribution.

Finally, we show that for all $\beta > \beta_c$, the mass $m(\beta)$ is real analytic.

The outline of the paper is as follows. In Sect. 3, we set up the Ornstein–Zernike analysis for self-avoiding walks, as introduced in [23]. In order to do this, we define two auxiliary generating functions: the cylinder function and the direct correlation function. The former is essentially a subadditive version of the two-point function, and can be shown to have the same exponential decay rate [23]. This allows us to establish basic properties of the walks. The direct correlation function is a restricted version of the cylinder function, obtained by summing over a subclass of walks obeying a local constraint, and is related to the cylinder function via an Ornstein–Zernike equation. Following the scheme of [23] (which is modeled on that of [10]), the Ornstein–Zernike analysis reduces the proof of the long-distance behavior to the conjecture that the direct correlation function has a strictly larger mass than that of the full cylinder function.

Sections 4 and 5 are the core of the paper. There we consider blocks of some fixed scale and define a generating function on these blocks which interpolates between the direct correlation function and the full cylinder function at that scale. This is done by restricting to walks which obey the local constraint of the direct correlation function over only part of the block. The idea is then to construct interpolating functions on any scale by patching together a sufficient number of block functions. Unfortunately, simple patching will not suffice due to the recurrence properties of the walks. We can, however, construct yet another generating function containing only the recurrent walks, and show that the rescaled interpolating function and the recurrent walk function are related by an *Ornstein–Zernike inequality*. Using straightforward estimates on the individual block functions and the recurrent walk function (in addition to certain observations on the properties of solutions to an Ornstein–Zernike equation), we obtain a bound on the mass of the rescaled interpolating function.

In Sect. 6, we use this bound to show that the direct correlation function has a strictly larger mass than the cylinder function, and hence that the latter has Ornstein–Zernike decay. Finally, we prove that similar results hold for the original two-point function. This is done via another convolution analysis which relates the pole structure of the transform of the two-point function to that of the cylinder function.

3. Basic Properties and the Ornstein–Zernike Criterion

In this section, we review some preliminary results from [23] which reduce the proof of Ornstein–Zernike behavior to showing that a certain nonperturbative criterion is satisfied. The first step is to find a subadditive function which has asymptotic properties similar to that of the two-point function. This is accomplished via

3.1. The Cylinder Generating Function. Among all walks which contribute to $G_{(L;a)}$ are a special class called *cylinder walks*, denoted by $\{w_H: 0 \to (L;a)\}$, which satisfy the restriction that every step of a w_H, save the first, has both endpoints in the region $1 \leqq x_1 \leqq L$. (In other words, the cylinder walks obey Dirichlet boundary conditions.) By this definition, the first step of a cylinder walk w_H: $0 \to (L;a)$ is along the bond between the origin and the point $(1;0)$. The cylinder generating functions are then defined by

$$H_{(L;a)}(\beta) = \sum_{w_H: 0 \to (L;a)} e^{-\beta |w_H|} \tag{3.1}$$

and

$$\mathbb{H}_L(\beta) = \sum_{a \in \mathbb{Z}^{d-1}} H_{(L;a)}(\beta). \tag{3.2}$$

The utility of the cylinder functions is that they obey a subadditive bound of the form

$$\mathbb{H}_{L_1+L_2} \geqq \mathbb{H}_{L_1} \mathbb{H}_{L_2}. \tag{3.3}$$

Using this, it is straightforward to establish the existence of a mass $m(\beta)$ with certain properties, as summarized below.

Proposition 3.1 ([23], Sect. 5.1). *$\forall \beta$, the limit*

$$m(\beta) \equiv \lim_{L \to \infty} [-L^{-1} \log \mathbb{H}_L(\beta)] \tag{3.4}$$

exists (in $\mathbb{R}^*$). Furthermore, m: $(0, \infty) \to \{-\infty \cup [0, \infty)\}$ is non-decreasing, concave and right continuous. Finally, uniformly in L, $m(\beta)$ provides the a priori bound

$$\mathbb{H}_L(\beta) \leqq e^{-m(\beta)L}. \tag{3.5}$$

The full two-point function has asymptotic decay similar to that of the cylinder function. Indeed, it is easy to show that the $\mathbb{G}_L$ obey the superadditive bound

$$\mathbb{G}_{L_1+L_2} \leqq \mathbb{G}_{L_1}\mathbb{G}_{L_2} \tag{3.6}$$

which is just the Simon inequality [28] for all self-avoiding walks to a $(d-1)$-dimensional hyperplane. From this, it follows that $\lim_{L\to\infty} [-L^{-1}\log\mathbb{G}_L(\beta)]$ exists, and that the $\mathbb{G}_L$ satisfy an a priori bound of the opposite form of (3.5). Finally, it is easy to show that if $\beta > \beta_c$, the mass of the $\mathbb{G}_L$ coincides with that of the $\mathbb{H}_L$. This follows from combining the a priori inequalities on $\mathbb{H}_L$ and $\mathbb{G}_L$ with the obvious bound

$$\mathbb{H}_L(\beta) \leqq \mathbb{G}_L(\beta) \leqq \chi^2(\beta)\mathbb{H}_L(\beta). \tag{3.7}$$

We have:

Proposition 3.2. $\forall\beta > \beta_c$,

$$\lim_{L\to\infty} [-L^{-1}\log\mathbb{G}_L(\beta)] = m(\beta), \tag{3.8}$$

where $m(\beta)$ is defined by (3.4). *Furthermore, uniformly in L,*

$$e^{-m(\beta)L} \leqq \mathbb{G}_L(\beta) \leqq \chi^2(\beta)e^{-m(\beta)L}. \tag{3.9}$$

Remark. The reader is cautioned that the elementary arguments presented above do *not* determine whether the self-avoiding walk has a divergent correlation length. In this regard, it is easy to establish that $\chi(\beta)$ diverges as $\beta\downarrow\beta_c$, reminiscent of a second order transition. (This can be done either with differential inequalities [29, 30], or by observing that the coefficients $N(n)$ in the expression $\chi(\beta) = \sum_n N(n)e^{-\beta n}$ satisfy $N(n) \geqq e^{\beta_c n}$.) However, in order to complete the analogy to a second order transition, one must also demonstrate the divergence of $m^{-1}(\beta)$ as $\beta\downarrow\beta_c$. The additional ingredient for self-avoiding walks is the existence ([23], Th. 5.6) of a function $P(m)$ with the property $P(m) < \infty \Leftrightarrow m > 0$, such that

$$\mathbb{G}_L \leqq P(m)e^{-mL}. \tag{3.10}$$

From this, one can establish $m(\beta)\downarrow 0$ as $\beta\downarrow\beta_c$, either by use of a Lieb–Simon inequality [28, 31], or directly from the observation that

$$\chi = \sum_{L\in\mathbb{Z}} \mathbb{G}_L \leqq 2P(m)/(1-e^{-m}). \tag{3.11}$$

The critical properties of the self-avoiding walk will play no role in our analysis.

Having determined the dominant exponential decay in Propositions 3.1 and 3.2, we are now in a position to examine the detailed asymptotic properties of the two-point function. To do this, we introduce

3.2. The Ornstein–Zernike Direct Correlation Function. Consider a cylinder walk $w_H: 0 \to (L; a)$. Let $n_j(w_H)$, $1 \leqq j \leqq L$, denote the number of times that the dual hyperplane $x_1 = j - \frac{1}{2}$ is pierced by the walk w_H. Note that since w_H is a cylinder walk, $n_j(w_H)$ is odd for every j, and by our boundary condition, $n_1(w_H) = 1$. We expect that generic walks will have many of the n_j's equal to 1; indeed, this is easy to verify for large β. The special walks for which this is forbidden, i.e. $\{w_H:0\to(L;a)|n_j(w_H)>1,\ \forall 1<j\leqq L\}$, are central to our analysis. The generating function defined by these walks is called the *direct correlation function*:

$$C_{(L;a)}(\beta)=\sum_{\substack{w_H\cdot 0\to(L,a)\\ n_j(w_H)>1,1<j\leqq L}} e^{-\beta|w_H|}, \tag{3.12}$$

$$\mathbb{C}_L(\beta)=\sum_{a\in\mathbb{Z}^{d-1}} C_{(L;a)}(\beta). \tag{3.13}$$

The utility of the $\mathbb{C}$ functions is that they are related to the $\mathbb{H}$'s via an *Ornstein–Zernike equation*:

$$\mathbb{H}_L=\sum_{N=0}^{L}\mathbb{C}_N\mathbb{H}_{L-N},\quad L\geqq 1. \tag{3.14a}$$

Here we have used the special conventions $\mathbb{H}_0=1$ and $\mathbb{C}_0=0$. More generally

$$H_{(L;a)}=\sum_{b\in\mathbb{Z}^{d-1}}\sum_{N=0}^{L} C_{(N;b)}H_{(L-N;a-b)},\quad L\geqq 1. \tag{3.14b}$$

It is precisely because of these equations that we can identify $\mathbb{C}$ as the direct correlation function.

Although the $\mathbb{C}$ functions are not subadditive, we may define their mass by

$$m_c(\beta)\equiv\liminf_{L\to\infty}[-L^{-1}\log\mathbb{C}_L(\beta)]. \tag{3.15}$$

Evidently $m_c(\beta)\geqq m(\beta)$. Indeed, it is easy to verify that $\lim_{\beta\to\infty}[m_c(\beta)/m(\beta)]=3$. (This follows from the fact that, for cylinder walks, $n_j(w_H)>1\Rightarrow n_j(w_H)=3$, and that both $\mathbb{C}$ and $\mathbb{H}$ concentrate near their minimum allowed walks as $\beta\to\infty$.)

The following theorem was derived in [10] in the context of a particular random surface model. An explicit proof for the S.A.W. (including a derivation of (3.14)) can be found in [23].

Theorem 3.3 ([23], Ths. 5.10 and 5.11). *Suppose $\beta>\beta_c$. Whenever $m_c(\beta)>m(\beta)$:*
(1) *there exist finite, positive constants $C_H(\beta)$ and $\Delta_H(\beta)$ such that*

$$|\mathbb{H}_L(\beta)e^{+m(\beta)L}-C_H(\beta)|\leqq e^{-\Delta_H(\beta)L};$$

(2) *there exist finite, positive constants $\alpha(\beta)$ and $\varepsilon_H(\beta)$ such that for all $a\in\mathbb{Z}^{d-1}$ satisfying $|a|<\varepsilon_H(\beta)L$,*

$$H_{(L;a)}(\beta)=C_H(\beta)[\alpha(\beta)\pi L]^{-(d-1)/2}e^{-m(\beta)L}e^{-a^2/\alpha(\beta)L}[1+O(L^{-1/2})];$$

(3) *$m(\beta)$ is real analytic.*

It follows from the discussion above that the condition $m_c(\beta)>m(\beta)$ is easily verified for large β; thus one can derive conclusions (1)–(3) in a perturbative regime. However, for our purposes, the significance of Theorem 3.3 is that it reduces the proof of Ornstein–Zernike behavior to the conjecture that $m_c(\beta)>m(\beta)\ \forall\beta>\beta_c$. The proof of this conjecture is the content of the next two and one-half sections. Eventually (in Sect. 6), we return to a consideration of noncylinder S.A.W.'s, and show that conclusions (1)–(3) hold for the full two-point function.

4. Construction of the Interpolating Functions

4.1. Control of the Large L Behavior of $\mathbb{C}_L$. The key ingredient in the proof of Theorem 3.3(1) is to show that whenever $m_c(\beta) > m(\beta)$, the Laplace transform $\mathbb{H}\tilde{}(z) \equiv \Sigma_L \mathbb{H}_L z^L$ has a simple pole at $z = e^{m(\beta)}$. Here, we wish to derive a weak converse to such a statement. Namely, given certain estimates on the function $\mathbb{H}\tilde{}(z)$, can these be used to get some control on the decay of the $\mathbb{C}_L$?

Our estimates on $\mathbb{H}\tilde{}(z)$ are provided by an analogue of (3.9):

$$\chi^{-2}(\beta)e^{-m(\beta)L} \leqq \mathbb{H}_L(\beta) \leqq e^{-m(\beta)L}, \tag{4.1}$$

which is a consequence of (3.7) and the a priori bounds on $\mathbb{G}_L$ and $\mathbb{H}_L$. It does *not* follow from (4.1) alone that $\mathbb{C}_L$ has a strictly larger decay rate than $\mathbb{H}_L$. Nevertheless, as shown below, (4.1) provides us with a moment condition on the quantities $\mathbb{C}_L e^{mL}$.

Proposition 4.1. *Suppose* $\beta > \beta_c$. *Then*

(i) $\sum_L \mathbb{C}_L e^{m(\beta)L} = 1,$

and

(ii) $\sum_L L\mathbb{C}_L e^{m(\beta)L} < \infty.$

Proof. Using the conventions $\mathbb{H}_0 \equiv 1$, $\mathbb{C}_0 \equiv 0$ and $\mathbb{C}_1 = \mathbb{H}_1$, we define the Laplace transforms

$$\mathbb{H}\tilde{}(z) \equiv \sum_{L=0}^{\infty} \mathbb{H}_L z^L \tag{4.2}$$

and

$$\mathbb{C}\tilde{}(z) \equiv \sum_{L=0}^{\infty} \mathbb{C}_L z^L. \tag{4.3}$$

By the Ornstein–Zernike equation (3.14a), these quantities are related via

$$\mathbb{H}\tilde{}(z) = [1 - \mathbb{C}\tilde{}(z)]^{-1}, \tag{4.4}$$

which is well-defined whenever $|ze^m| < 1$.

Let us study $\mathbb{C}\tilde{}(z)$ as a function of the real variable $x \in (0, e^m)$. Since all the coefficients $\mathbb{C}_L$ in (4.3) are real and positive, we may express the left sides of (i) and (ii) as the $x \uparrow e^m$ limits of $\mathbb{C}\tilde{}(x)$ and $x(d/dx)\mathbb{C}\tilde{}(x)$, respectively. The latter quantities may be calculated via (4.4) and the bound

$$\chi^{-2}(\beta)[1 - xe^{-m}]^{-1} \leqq \mathbb{H}\tilde{}(x) \leqq [1 - xe^{-m}]^{-1}, \tag{4.5}$$

which follows from summing (4.1) over L. In particular, (4.4) and (4.5) imply

$\sum_L \mathbb{C}_L e^{m(\beta)L} = \mathbb{C}^{\sim}(e^m) = 1$, which proves (i). To establish (ii), observe that

$$x\frac{d}{dx}\mathbb{C}^{\sim}(x) = x[\mathbb{H}^{\sim}(x)]^{-2}\frac{d}{dx}\mathbb{H}^{\sim}(x) \leqq xe^{-m(\beta)}\chi^4(\beta), \tag{4.6}$$

so that $\sum_L L\mathbb{C}_L e^{m(\beta)L} \leqq \chi^4(\beta) < \infty$ for $\beta > \beta_c$. ∎

4.2 The Block Variables. As indicated in the previous subsection, analytical methods alone do not give the necessary control on the decay rate of $\mathbb{C}_L$. The first step in circumventing this difficulty is to define block variables which interpolate between $\mathbb{H}_L$ and $\mathbb{C}_L$. To this end, take A and R to be positive integers and let $L = 2A + R$. We divide the (cylinder) region $0 \leqq x_1 \leqq L$ into three strips specified by their x_1 coordinates: $S_l = \{x_1 \in \mathbb{Z} | 0 \leqq x_1 \leqq A\}$, $S_c = \{x_1 \in \mathbb{Z} | A < x_1 < A + R\}$, and $S_r = \{x_1 \in \mathbb{Z} | A + R \leqq x_1 \leqq 2A + R\}$. Our block walks are required to obey the "$\mathbb{C}$ condition" (i.e., $n_j(w) > 1$) in the central strip:

$$\mathfrak{B}_1^*(L, R) = \bigcup_{a \in \mathbb{Z}^{d-1}} \{w_H : 0 \to (L; a) | \forall j \in S_c \; n_j(w_H) > 1\}. \tag{4.7}$$

The corresponding generating function is given by

$$\mathbb{B}_1^*(L, R; \beta) = \sum_{w \in \mathfrak{B}_1^*(L, R)} e^{-\beta|w|}. \tag{4.8}$$

It is also convenient to define modified block walks which need not obey the cylinder restriction on the left boundary. To be explicit, let $\{w_J : 0 \to (L; a)\}$ denote the set of self-avoiding walks from the origin to $(L; a)$, with every step in the region $x_1 \leqq L$. (Thus, the "J walks" obey Dirichlet conditions only on the right boundary.) One could define a corresponding generating function $\mathbb{J}_L(\beta)$; however, for our purposes, it suffices to note that $\mathbb{J}_L(\beta) \leqq \mathbb{G}_L(\beta)$. The modified block walks are those J walks which obey the $\mathbb{C}$ condition in the central region:

$$\mathfrak{B}_1(L, R) = \bigcup_{a \in \mathbb{Z}^{d-1}} \{w_J : 0 \to (L; a) | \forall j \in S_c \; n_j(w_J) > 1\}, \tag{4.9}$$

so that

$$\mathbb{B}_1(L, R; \beta) = \sum_{w \in \mathfrak{B}_1(L, R)} e^{-\beta|w|}. \tag{4.10}$$

Obviously, $\mathbb{B}_1(L, R)$ and $\mathbb{B}_1^*(L, R)$ provide upper bounds on $\mathbb{C}_L$. The content of the following lemma is that, for R large enough, the block functions are small relative to $\mathbb{H}_L$.

Lemma 4.2. *Suppose $\beta > \beta_c$ and $L > 3R$. Then, uniformly in L, there exists a $\delta(R)$ with $\lim_{R \to \infty} \delta(R) = 0$ such that*

$$\mathbb{B}_1^*(L, R) \leqq \mathbb{B}_1(L, R) \leqq \delta(R) e^{-mL}.$$

Proof. First let us derive a bound of the desired form for the function $\mathbb{B}_1^*$. To this end, we write an exact expression for $\mathbb{B}_1^*$ and use the a priori bound on $\mathbb{H}$:

$$\mathbb{B}_1^*(L,R;\beta) = \sum_{N=1}^{A}\sum_{K=0}^{A} \mathbb{H}_{A-N}\mathbb{C}_{R+N+K}\mathbb{H}_{A-K} \leqq e^{-mL}\sum_{N=1}^{\infty}\sum_{K=0}^{\infty} \mathbb{C}_{R+N+K}\,e^{+m[R+N+K]} \tag{4.11}$$

Next, we observe that for any nonnegative function F:

$$\sum_{N=1}^{\infty}\sum_{K=0}^{\infty} F(N+K) = \sum_{n=0}^{\infty} nF(n), \tag{4.12}$$

so that (4.11) implies

$$\mathbb{B}_1^*(L,R;\beta) \leqq e^{-mL}\sum_{N=1}^{\infty}\sum_{n=0}^{\infty} n\mathbb{C}_{R+n}e^{+m[R+n]}. \tag{4.13}$$

Now notice that the coefficient of e^{-mL} in (4.13) is bounded above by $\sum_{n\geqq R} n\mathbb{C}_n e^{+mn}$. Since this is the *tail* of a convergent sum (cf. Prop. 4.1 (ii)), it tends to zero as $R\uparrow\infty$.

In order to derive a similar bound for $\mathbb{B}_1(L,R;\beta)$, we must account for those walks which journey into the region $x_1<0$. Most of these can be handled by replacing $\mathbb{H}_{A-N}$ by $\mathbb{G}_{A-N}$ in the bound (4.11), which modifies the final bound by a multiplicative factor of no more than $\chi^2(\beta)$. This, however, does not take into account the (relatively rare) walks which visit the region $x_1<0$ *after* having visited S_c. The contribution of such walks can be bounded above by the additional term

$$\mathbb{G}_A^2\mathbb{G}_L \leqq e^{-mL}\chi^6 e^{-2mA}. \tag{4.14}$$

Since by assumption $A>R$, the coefficient of e^{-mL} tends to zero as $R\uparrow\infty$. This gives the desired result with

$$\delta(R) \leqq \chi^2\Big[\sum_{n\geqq R} n\mathbb{C}_n e^{+mn} + \chi^4 e^{-2mR}\Big]. \tag{4.15}$$

■

4.3. The Rescaled Variables. In order to generalize the previous construction to arbitrary scales, we choose $N\geqq 1$ and consider walks $w_H\colon 0\to(NL;a)$ which satisfy the "$\mathbb{C}$ condition" in each of the central strips, with x_1 coordinates

$$S_c(j) = \{x_1\in\mathbb{Z}\,|\,A+(j-1)L < x_1 < A+(j-1)L+R\}, \tag{4.16}$$

$1\leqq j\leqq N$. We denote this class of walks by

$$\mathfrak{B}_N^*(L,R) = \bigcup_{a\in\mathbb{Z}^{d-1}} \{w_H\colon 0\to(NL;a)\,|\,\forall 1\leqq i\leqq N\ \forall j\in S_c(i)\ n_j(w_H)>1\}, \tag{4.17}$$

and the corresponding generating function by

$$\mathbb{B}_N^*(L,R;\beta) = \sum_{w\in\mathfrak{B}_N^*(L,R)} e^{-\beta|w|}. \tag{4.18}$$

It is easy to see that these rescaled interpolating functions obey the subadditive bound

$$\mathbb{B}_{N+K}^* \geqq \mathbb{B}_N^*\mathbb{B}_K^*, \tag{4.19}$$

so that the limit $m_B(L,R:\beta) \equiv \lim_{N\to\infty} [-N^{-1} \log \mathbb{B}_N^*(L,R;\beta)]$ exists and provides the usual a priori inequality

$$\mathbb{B}_N^* \leqq e^{-m_B(L,R)N}. \tag{4.20}$$

Again, it is convenient to define modified interpolating functions by summing over walks which may enter the region $x_1 < 0$. Thus we denote by $\mathfrak{B}_N(L,R)$ the set of walks defined as in (4.17), with w_H replaced by w_J. The corresponding generating function will be denoted by $\mathbb{B}_N(L,R)$.

Notice that the rescaled functions defined here provide a bound on the direct correlation function:

$$\mathbb{B}_N(L,R) \geqq \mathbb{B}_N^*(L,R) \geqq \mathbb{C}_{NL}. \tag{4.21}$$

5. An Ornstein–Zernike Inequality for the Rescaled Variables

Consider the walks contributing to the rescaled interpolating function $\mathbb{B}_N^*$. Since these include all walks which can be obtained by patching together block walks in $\mathfrak{B}_1^*$, we have the obvious bound

$$\mathbb{B}_N^*(L,R) \geqq [\mathbb{B}_1^*(L,R)]^N. \tag{5.1}$$

Indeed, for any finite L and R, the inequality (5.1) is strict.

Part of the difference between the two sides of (5.1) is due to walks which have small fluctuations at the edges between successive blocks. These can be taken into account by simply relaxing the restriction that a block walk not enter the previous block, i.e. by replacing $\mathbb{B}_1^*(L,R)$ with $\mathbb{B}_1(L,R)$. Thus one might hope to obtain an inequality of the form $\mathbb{B}_N^*(L,R) \leqq [\mathbb{B}_1(L,R)]^N$ to complement (5.1). Were such an inequality legitimate, we would be done. Indeed, by Lemma 4.2, such an inequality would imply $m_B(L,R;\beta) > m(\beta)L$ for suitably chosen R and L. This, combined with (4.21), would in turn imply[1] $m_c(\beta) > m(\beta)$, the desired result.

However, it is clear that the inequality $\mathbb{B}_N^*(L,R) \leqq [\mathbb{B}_1(L,R)]^N$ is not quite correct. Consider, for example, walks which have satisfied the $\mathbb{C}$ condition in $S_c(2)$ *before* satisfying it in $S_c(1)$. While such walks can be found in $\mathfrak{B}_2^*$, it is not possible to construct all of them by patching together two walks in $\mathfrak{B}_1$.

As the above example illustrates, the "missing walks" are those which travel back to the first block after having wandered a substantial distance from the origin. They are thus related to the *recurrent walks* which are expected to produce deviations from (short-distance) Ornstein–Zernike behavior at the critical point in low dimension. Clearly, these are the walks we must control. We do this by defining a generating function for these walks and showing that it plays the role of an approximate direct correlation function for the generating function $\mathbb{B}_N$.

5.1. The Recurrent Walk Generating Function. Let $w \in \mathfrak{B}_N(L,N)$, $N > 1$, and denote by $w^\dagger$ the walk obtained by amputating w at its first point of intersection with the

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[1] Even this implication requires some further argument since m_c is only defined as an inferior limit. It is, however, straightforward (cf. Theorem 6.1) to obtain the result $m_c > m$ from $m_B/L > m$

plane $x_1 = (N-1)L$. As explained above, it need *not be the case that* $n_j(w^\dagger)$ exceeds 1 for all $j \in S_c(1)$, since the full walk w may correct this deficiency after having visited the final block. Indeed, these are precisely the recurrent walks we wish to control. We thus define

$$\mathscr{V}_N^\wedge(L,R) = \{w \in \mathfrak{B}_N(L,R) \mid \exists j \in S_c(1) \text{ s.t. } n_j(w^\dagger) = 1\}, \tag{5.2}$$

and the generating function

$$\mathbb{V}_N^\wedge(L,R;\beta) = \sum_{w \in \mathscr{V}_N^\wedge(L,R)} e^{-\beta|w|}. \tag{5.3}$$

For technical reasons, it is useful to define a slightly modified version of the set $\mathscr{V}_N^\wedge(L,R)$ which includes also walks that need not satisfy the $\mathbb{C}$ condition in the final block:

$$\mathscr{V}_N(L,R) = \bigcup_{a \in \mathbb{Z}^{d-1}} \{w_J : 0 \to (NL;a) \mid \forall 1 \leqq i \leqq N-1 \; \forall j \in S_c(i)\, n_j(w_J) > 1$$

and

$$\exists k \in S_c(1) \text{ s.t. } n_k(w_J^\dagger) = 1\}. \tag{5.4}$$

Our *recurrent walk generating function* is given by

$$\mathbb{V}_N(L,R;\beta) = \sum_{w \in \mathscr{V}_N(L,R)} e^{-\beta|w|}. \tag{5.5}$$

Notice that $\mathscr{V}_N(L,R)$ includes all walks which are in $\mathscr{V}_N^\wedge(L,R)$, so that

$$\mathbb{V}_N^\wedge(L,R) \leqq \mathbb{V}_N(L,R). \tag{5.6}$$

The following lemma shows that, for $\beta > \beta_c$, the recurrent walk generating function decays much more quickly than $\mathbb{H}_{NL}$.

Lemma 5.1. *Suppose $\beta > \beta_c$ and $N > 1$. Then, uniformly in N and L, there exists an $\Omega(R) < \infty$ such that*

$$\mathbb{V}_N(L,R) \leqq \Omega(R) e^{-3mL(N-1)}.$$

Proof. Any walk in $\mathscr{V}_N(L,R)$ must travel to the plane $x_1 = (N-1)L$, return to the plane $x_1 = L - A$, and then travel to the plane $x_1 = NL$. Having done so, it automatically satisfies the $\mathbb{C}$ condition in all but the two boundary blocks. Relaxing the constraint that the walk satisfy the $\mathbb{C}$ condition in $S_c(1)$ after having returned to the first block, and allowing the various pieces mentioned above to intersect, we have the obvious bound

$$\mathbb{V}_N(L,R) \leqq \mathbb{G}_{(N-1)L} \mathbb{G}_{(N-2)L+A} \mathbb{G}_{(N-1)L+A} \leqq \chi^6 e^{+mR} e^{-3(N-1)mL}, \tag{5.7}$$

which is the desired result with $\Omega(R) \leqq \chi^6 e^{+mR}$. ∎

5.2. The Ornstein–Zernike Inequality. Although the recurrent walk function does not allow us to write an exact convolution expression for the $\mathbb{B}_N$, it does enable us to derive an Ornstein–Zernike inequality. In the following proposition, we demonstrate the utility of such an inequality.

Proposition 5.2. *Let $\{f_n\}$ and $\{g_n\}$ be real nonnegative sequences with $f_0 = 1$, $g_0 = 0$*

and $f_1 = g_1$, *and suppose that these sequences satisfy an Ornstein–Zernike inequality for* $n \geqq 1$:

$$f_n \leqq \sum_{k=0}^{n} g_k f_{n-k}. \tag{5.8}$$

(i) *If the sequence* $\{F_n\}$ *with* $F_0 = 1$ *is a solution to the corresponding Ornstein–Zernike equality for* $n \geqq 1$, i.e.

$$F_n = \sum_{k=0}^{n} g_k F_{n-k}. \tag{5.9}$$

then the F_n *provide an upper bound on the* f_n:

$$F_n \geqq f_n.$$

(ii) *If the* g_n, $n \geqq 1$, *are replaced by the upper bounds* $\Gamma_n \geqq g_n$ *and if* $\Gamma_0 = 1$, *then the* Φ_n *defined as a solution to the new Ornstein–Zernike equality for* $n \geqq 1$, i.e.

$$\Phi_n = \sum_{k=0}^{n} \Gamma_k \Phi_{n-k}. \tag{5.10}$$

with $\Phi_0 = 1$, *provides an upper bound on the* F_n:

$$\Phi_n \geqq F_n.$$

Proof. The proposition follows easily by induction on n. One need only note that the solutions, F_n and Φ_n, to the Ornstein–Zernike Eqs. (5.9) and (5.10) are determined recursively from a knowledge of the zeroth terms. ∎

In the following lemma, we use an Ornstein–Zernike inequality to obtain our key estimate on separation of the masses.

Lemma 5.3. *Suppose* $\beta > \beta_c$. *Then for suitable choice of* L *and* R,

$$m_B(L, R; \beta) \geqq m(\beta)L.$$

Proof. Consider the walks which comprise $\mathfrak{B}_N(L, R)$, $N > 1$. These may be classified according to the maximum x_1 coordinate (in units of L) that the walk has achieved before the $\mathbb{C}$ condition is satisfied in $S_c(1)$. For example, those walks which have travelled all the way to the N^{th} block before returning to satisfy the $\mathbb{C}$ condition in $S_c(1)$ contribute exactly $\hat{\mathscr{V}}_N(L, R)$ to $\mathfrak{B}_N(L, R)$.

We claim that the walks which go *only* as far as the K^{th} block before returning to satisfy the $\mathbb{C}$ condition in $S_c(1)$ yield a contribution to $\mathbb{B}_N$ which does not exceed $\mathbb{V}_K \mathbb{B}_{N-K}$. Indeed, any such walk can be decomposed into two pieces: The first is a walk which "turns around" somewhere in the K^{th} block, returns to satisfy the $\mathbb{C}$ condition in $S_c(1)$ (thereby also satisfying it in $S_c(2), \ldots, S_c(K-1)$), and finally travels to the plane $x_1 = KL$. Note that this walk need not satisfy the $\mathbb{C}$ condition in $S_c(K)$ before reaching the plane $x_1 = KL$; hence the factor $\mathbb{V}_K$, rather than $\hat{\mathbb{V}}_K$. The second walk travels from the plane $x_1 = KL$ to $x_1 = NL$, eventually satisfying the $\mathbb{C}$ condition in $S_c(K+1), \ldots, S_c(N)$, and perhaps also satisfying it in $S_c(K)$ by travelling into the region $x_1 < KL$. Ignoring possible violations of the $\mathbb{C}$ condition in the K^{th} block, and relaxing the constraint that the two pieces mentioned above not intersect yields the desired expression.

By the above reasoning, $\forall N > 1$ we have

$$\mathbb{B}_N \leqq \mathbb{V}_N^{\hat{}} + \sum_{K=0}^{N-1} \mathbb{V}_K \mathbb{B}_{N-K} \leqq \mathbb{V}_N + \sum_{K=0}^{N-1} \mathbb{V}_K \mathbb{B}_{N-K}. \tag{5.11}$$

In this expression, we have set the convention $\mathbb{V}_1 \equiv \mathbb{B}_1$, since the recurrent walk function was only defined for $N > 1$. If we supplement this with the conventions $\mathbb{B}_0 \equiv 1$ and $\mathbb{V}_0 \equiv 0$, (5.11) becomes

$$\mathbb{B}_N \leqq \mathbb{V}_N + \sum_{K=0}^{N} \mathbb{V}_K \mathbb{B}_{N-K} \quad \forall N \geqq 1, \tag{5.12}$$

which is our Ornstein–Zernike inequality.

By Prop. 5.2, any solution to (5.12) as an *equality* provides an upper bound on the $\mathbb{B}_N$. Moreover, this bound only deteriorates if we replace $\mathbb{V}_1 \equiv \mathbb{B}_1$ by the upper bound of Lemma 4.2, and $\mathbb{V}_N$, $N \geqq 2$, by the upper bound of Lemma 5.1.

Let us therefore analyze the asymptotics of the equation

$$b_N \leqq v_N + \sum_{K=0}^{N} v_K b_{N-K}, \quad N \geqq 1, \tag{5.13}$$

with $b_0 = 1$, $v_0 = 0$, $b_1 = v_1 = \delta(R)\alpha(L)$ and, for $K \geqq 2$, $v_K = \Omega(R)\alpha(L)^{3(K-1)}$. Here $\alpha(L) \equiv e^{-mL}$. To do this, we define the Laplace transforms $b^{\sim}(z) = \sum_{N \geqq 0} b_N z^N$ and $v^{\sim}(z) = \sum_{N \geq 0} v_N z^N$. Transforming (5.13), we obtain

$$b^{\sim}(z) = [1 - v^{\sim}(z)]^{-1} = [1 - \delta\alpha z - \Omega\alpha^3 z^2/(1 - \alpha^3 z)]^{-1}. \tag{5.14}$$

The asymptotic (i.e., exponential) behavior of b_N is determined by the smallest real solution to

$$v^{\sim}(z_0) = 1. \tag{5.15}$$

In our case, this amounts simply to the solution of a quadratic equation. Neglecting the nonlinear piece in (5.14), the solution is just

$$z_0(1) = (\delta\alpha)^{-1}. \tag{5.16}$$

A more refined analysis easily verifies that the above solution is quite accurate provided that

$$\text{(i)}\ \delta \gg \alpha^2, \tag{5.17a}$$

$$\text{(ii)}\ \Omega/\delta \gg \alpha, \tag{5.17b}$$

and

$$\text{(iii)}\ \delta^2/\Omega \gg \alpha, \tag{5.17c}$$

which can be arranged by choosing $\alpha = e^{-mL}$ small enough. Of course, since $\log \delta$ will provide our correction to the mass, we would like to take $\delta < 1$. Thus we first choose R such that $\delta = \delta(R) < 1$, which also fixes $\Omega = \Omega(R)$. Then L is chosen as large as necessary to ensure that (i)–(iii) are satisfied. We thereby obtain

$$e^{m_B(L,R)} \geqq \delta^{-1}(R) e^{mL}[1 - (\Omega/4\delta^2)e^{-mL} + O(e^{-2mL})], \tag{5.18}$$

which is the desired result, i.e.

$$m_B(L,R) \geqq mL + |\log\delta| + O(e^{-mL}). \tag{5.19}$$

6. Ornstein–Zernike Behavior of the Generating Functions

In this section, we establish that the cylinder and full generating functions exhibit Ornstein–Zernike behavior at all noncritical temperatures. We first treat the cylinder function, for which the result follows fairly easily from the analysis of the previous section. The full generating function, which requires somewhat more care, is examined in the second part of this section.

6.1. The Cylinder Function.

Theorem 6.1. *For every* $\beta > \beta_c$,

$$m_c(\beta) > m(\beta).$$

This of course implies:

Corollary. *Conclusions* (1), (2) *and* (3) *of Theorem 3.3 hold for every* $\beta > \beta_c$.

Proof of Theorem 6.1. By Lemma 5.2, we may choose L and R such that $m_B(L,R) \geqq mL$. As explained at the beginning of Sect. 5, the inequality $m_c > m$ would follow immediately from this and the obvious bound $\mathbb{B}_N^*(L,R) \geqq \mathbb{C}_{NL}$ *if* we could assert that $m_c L = \liminf_{N\to\infty}[-N^{-1}\log\mathbb{C}_{NL}]$.

This minor annoyance can be circumvented in a number of ways. For example, consider another class of cylinder walks $\mathfrak{B}_K^{\square}(L,R)$ which agrees with $\mathfrak{B}_K^*(L,R)$ if $K = NL$ for any positive integer N; otherwise, the walks in $\mathfrak{B}_K^{\square}(L,R)$ satisfy the $\mathbb{C}$ condition in $S_c(1),\ldots,S_c(N(K))$, where $N(K)$ is the largest integer smaller than K/L. Let us denote the corresponding generating function by $\mathbb{B}_K^{\square}(L,R)$. It is obvious that $\mathbb{C}_K \leqq \mathbb{B}_K^{\square}\ \forall K$. Thus $m_c L \geqq M_B(L,R) \equiv \liminf_{K\to\infty}[-K^{-1}\log\mathbb{B}_K^{\square}(L,R)]$.

It suffices to show that $M_B(L,R) \geqq mL$. To this end, consider the walks in $\mathfrak{B}_K^{\square}(L,R)$. The contribution of those walks which satisfy the $\mathbb{C}$ condition in every region $S_c(i)$, $1 \leqq i \leqq N(K)$, *before* reaching the plane $x_1 = N(K)L$ may be bounded above by $\chi\mathbb{B}_N^*(L,R)$ with $N = N(K)$. Otherwise, there is an earliest block $P < N(K)$ in which the $\mathbb{C}$ condition is not satisfied before the walk reaches $x_1 = N(K)L$. It is easy to verify that the contribution of these walks is no more than $\chi\mathbb{B}_{N-P}^*\mathbb{G}_{PL}(\mathbb{G}_{(P-1)L+A})^2$. We have

$$\mathbb{B}_K^{\square} \leqq \chi\mathbb{B}_N^* + \chi\sum_{p=1}^{N-1}\mathbb{B}_{N-P}^*\mathbb{G}_{PL}(\mathbb{G}_{(P-1)L+A})^2. \tag{6.1}$$

Using the upper bound (3.9) on $\mathbb{G}$ and the a priori bound (4.20) on $\mathbb{B}^*$, this implies

$$\mathbb{B}_K^{\square} \leqq \chi^7 N e^{+m(L+R)}e^{-\mu N} \tag{6.2}$$

with $\mu \equiv \min\{m_B, 3mL\}$. Thus

$$m_c \geqq M_B/L \geqq \mu/L = \min\{m_B/L, 3m\} > m, \tag{6.3}$$

which is the desired result. ■

6.2. The Full Generating Function. Here we extend the results of the previous subsection to the full generating functions $\mathbb{G}_L$ and $G_{(L;a)}$. The strategy is to express $\mathbb{G}\check{}(z)$ in terms of $\mathbb{H}\check{}(z)$ and show that, in the vicinity of $z=e^m$, the required modifications are analytic. This is done via a convolution analysis for the $\mathbb{G}_L$.

The first step is to define an analogue of $\mathbb{C}_L$ for walks which need not obey a cylinder restriction. Thus consider walks $w:0\to(L;a)$ which satisfy $n_j(w)=0$ or $n_j(w)>1$ for every $j\in\mathbb{Z}$. We denote the generating functions for such walks by $K_{(L;a)}(\beta)$ and $\mathbb{K}_L(\beta)=\sum_a K_{(L;a)}(\beta)$.

It is also convenient to have a direct correlation function for walks $w:0\to(L;a)$ which obey a cylinder restriction only on the right boundary $x_1=L$. We thus consider that subset of the walks contributing to $K_{(L;a)}(\beta)$ which satisfy the additional restriction $n_j(w)=0$ for every $j>L$. The corresponding generating functions will be denoted by $D_{(L;a)}(\beta)$ and $\mathbb{D}_L(\beta)=\sum_a D_{(L;a)}(\beta)$.

As with the $\mathbb{C}$ functions, we may define the masses of $\mathbb{K}$ and $\mathbb{D}$ via inferior limits:

$$m_K(\beta)\equiv\liminf_{L\to\infty}[-L^{-1}\log\mathbb{K}_L(\beta)], \tag{6.4}$$

$$m_D(\beta)\equiv\liminf_{L\to\infty}[-L^{-1}\log\mathbb{D}_L(\beta)]. \tag{6.5}$$

By an analysis along the lines of Th. 6.1 (cf. Eqs. (6.1)–(6.3)), it is straightforward to show that these masses are strictly larger than m:

Proposition 6.2. *Let* $\mu_c(\beta)\equiv\min\{m_c(\beta),3m(\beta)\}$. *For every* $\beta>\beta_c$,

$$m_K(\beta)\geqq\mu_c(\beta)>m(\beta)$$

and

$$m_D(\beta)\geqq\mu_c(\beta)>m(\beta).$$

Thus the $\mathbb{K}$ and $\mathbb{D}$ functions retain the essential property of the direct correlation function. This enables us to establish our principal result:

Theorem 6.3. *For every* $\beta>\beta_c$:

(1) *there exist finite, positive constants* $C(\beta)$ *and* $\Delta(\beta)$ *such that*

$$|\mathbb{G}_L(\beta)e^{+m(\beta)L}-C(\beta)|\leqq e^{-\Delta(\beta)L};$$

(2) *there exist finite, positive constants* $\alpha(\beta)$ *and* $\varepsilon(\beta)$ *such that for all* $a\in\mathbb{Z}^{d-1}$ *satisfying* $|a|<\varepsilon(\beta)L$,

$$G_{(L;a)}(\beta)=C(\beta)[\alpha(\beta)\pi L]^{-(d-1)/2}e^{-m(\beta)L}e^{-a^2/\alpha(\beta)L}[1+O(L^{-1/2})];$$

(3) $m(\beta)$ *is real analytic.*

Proof. Analyticity of $m(\beta)$ has already been established in the corollary to Th. 6.1, since the mass of the full generating function is identical to that of the cylinder function.

In order to prove (1) and (2), we perform another convolution analysis. To this end, consider all walks which contribute to $\mathbb{G}_L$. Each such walk either has no point j for which $n_j(w) = 1$ or it has at least one. The contribution of the former walks is just $\mathbb{K}_L$. In the latter case, we may locate the outermost "pair", $j_l = N + 1$ and $j_r = L - P$, of points for which $n_j(w) = 1$. (We of course include the special case $N + 1 = L - P$ of a degenerate pair.) It is easily seen that the contribution of such walks is exactly $\mathbb{D}_N \mathbb{H}_{L-P-(N+1)}\, e^{-\beta} \mathbb{D}_P$. We thus have the identity

$$\mathbb{G}_L = \mathbb{K}_L + e^{-\beta} \sum_{\substack{N \geqq 0 \\ N+P \leqq L-1}} \sum_{P \geqq 0} \mathbb{D}_N \mathbb{D}_P \mathbb{H}_{L-P-(N+1)} \tag{6.6}$$

with the convention $\mathbb{H}_0 = 1$. Transforming, this becomes

$$\mathbb{G}^{\sim}(z) = \mathbb{K}^{\sim}(z) + z e^{-\beta} [\mathbb{D}^{\sim}(z)]^2 \mathbb{H}^{\sim}(z). \tag{6.7}$$

It is now straightforward to verify (1) and (2). For example, to establish (1), note that (6.7) and Proposition 6.2 imply that $\mathbb{G}^{\sim}(z)$ may be written in the form

$$\mathbb{G}^{\sim}(z) = g(z) \mathbb{H}^{\sim}(z). \tag{6.8}$$

where $g(z)$ is analytic in some disk $|z| < R_g$, with $R_g > e^m$. From our analysis of the cylinder functions, we already know that $\mathbb{H}^{\sim}(z)$ has a simple pole at $z = e^m$ and no other poles in some larger region, so that it may be written in the form $\mathbb{H}^{\sim}(z) = h(z)[1 - ze^{-m}]^{-1}$ with $h(z)$ analytic in a disk of radius $R_h > e^m$. Thus

$$\mathbb{G}^{\sim}(z) = g(z) h(z) [1 - ze^{-m}]^{-1}, \tag{6.9}$$

where gh is analytic in a disk of radius $R = \min\{R_g, R_h\}$. Given (6.9), one need only apply the Cauchy bounds to show that $\mathbb{G}_L$ converges *exponentially* to $C(\beta)e^{-mL}$ for some $C(\beta) < \infty$ (see [23], Eqs. (5.30)–(5.32)). Note, however, that both $C(\beta)$ and the rate of exponential convergence are determined by the coefficients in the power series expansion of gh, so that these "constants" will differ from those of the cylinder function.

It is similarly straightforward (but tedious) to establish (2) by following the analogous derivation for the cylinder function (see [23], Th. 5.10). One need only note that the strictly larger masses of $\mathbb{K}$ and $\mathbb{D}$ imply analyticity of the residues. Again, the constant $\varepsilon(\beta)$ may differ from that for the cylinder function. ∎

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References

1. Ornstein, L. S., Zernike, F.: Proc. Acad. Sci. (Amst.) **17**, 793 (1914)
2. Abraham, D. B., Kunz, H.: Ornstein–Zernike theory of classical fluids at low density. Phys. Rev. Lett. **39**, 1011 (1977)

3. Paes-Leme, P. J.: Ornstein–Zernike and analyticity properties for classical lattice spin systems. Ann. Phys. (N.Y.) **115**, 367 (1978)
4. Schor, R. S.: The particle structure of ν-dimensional Ising models at low temperatures. Commun. Math. Phys. **59**, 213 (1978)
5. Simon, B.: The rate of falloff of Ising model correlations at large temperature. J. Stat. Phys. **26**, 53 (1981)
6. Schor, R. S.: Existence of glueballs in strongly coupled lattice gauge theories. Nucl. Phys. B. **222**, 71 (1983)
7. Schor, R. S.: Excited glueball states on a lattice from first principles. Phys. Lett. B **132**, 161 (1983)
8. Schor, R. S.: Glueball spectroscopy in strongly coupled lattice gauge theories. Commun. Math. Phys. **92**, 369 (1984)
9. Abraham, D. B., Chayes, J. T., Chayes, L.: Statistical mechanics of lattice tubes. Phys. Rev. D **30**, 841 (1984)
10. Abraham, D. B., Chayes, J. T., Chayes, L.: Random surface correlation functions. Commun. Math Phys. **96**, 439 (1984)
11. Abraham, D. B., Chayes, J. T., Chayes, L.: Nonperturbative analysis of a model of random surfaces. Nucl. Phys. B **251**, [FS **13**], 553 (1985)
12. Bricmont, J., Fröhlich, J.: An order parameter distinguishing between different phases of lattice gauge theories with matter fields. Phys. Lett. B **122**, 73 (1983)
13. Bricmont, J., Fröhlich, J.: Statistical mechanical methods in particle structure analysis of lattice field theories. Part 1: General results. Nucl. Phys. B **251**, [FS **13**], 517 (1985)
14. Bricmont, J., Fröhlich, J.: Statistical mechanical methods in particle structure analysis of lattice field theories. Part II: Scalar and surface models. Commun. Math. Phys. **98**, 553 (1985)
15. O'Carroll, M.: Analyticity properties and a convergent expansion for the inverse correlation length of the high-temperature d-dimensional Ising model. J. Stat. Phys. **34**, 597 (1984)
16. O'Carroll, M., Barbosa, W. D.: Analyticity properties and a convergent expansion for the inverse correlation length of the low-temperature d-dimensional Ising model. J. Stat. Phys. **34**, 609 (1984)
17. O'Carroll, M.: A convergent perturbation theory for particle masses in "Euclidean" lattice quantum field theory applied to the d-dimensional Ising model. Phys. Lett. B **143**, 188 (1984)
18. O'Carroll, M., Braga, G.: Analyticity properties and a convergent expansion for the glueball mass and dispersion curve of strongly coupled Euclidean $2+1$ lattice gauge theories. J. Math. Phys. **25**, 2741 (1984)
19. O'Carroll, M., Braga, G., Schor, R. S.: Mass spectrum and mass splitting in $2+1$ strongly coupled lattice gauge theories (Preprint)
20. O'Carroll, M., Braga, G.: Convergent expansions for glueball masses in $3+1$ strongly coupled lattice gauge theories (Preprint)
21. O'Carroll, M.: Convergent expansions for excited glueball masses in $2+1$ strongly coupled lattice gauge theories. (Preprint)
22. Chayes, J. T., Chayes, L.: Random tubes as a model of pair correlations. Contemp. Math. **41**, 11 (1985)
23. Chayes, J. T., Chayes, L.: Percolation and random media. In: Critical phenomena, random systems and gauge theories, Les Houches, Session, XLIII, K. Osterwalder and R. Stora, (eds) Amsterdam, Oxford, New York: Elsevier (1986)
24. Fisher, M. E.: Correlation functions and the critical region of simple fluids. J. Math. Phys. **5**, 944 (1964)
25. Aizenman, M.: Geometric analysis of φ^4 fields and Ising models. Parts 1 and II. Commun. Math. Phys. **86**, 1 (1982)
26. Fröhlich, J.: On the triviality of $\lambda\varphi_d^4$ theories and the approach to the critical point in $d>4$ dimensions. Nucl. Phys. B **200**, [FS4], 281 (1982)
27. Hammersley, J. M.: Percolation processes. II. The connective constant. Proc. Camb. Phil. Soc. **53**, 642 (1957)
28. Simon, B.: Correlation inequalities and decay of correlations in ferromagnets. Commun. Math. Phys. **77**, 111 (1980)

29. Aizenman, M., Newman, C. M.: Tree graph inequalities and critical behavior in percolation models. J. Stat. Phys. **36**, 107 (1984)
30. Bovier, A., Felder, G., Fröhlich, J.: On the critical properties of Edwards and self-avoiding walk models of polymer chains. Nucl. Phys. B **230** [FS **10**], 119 (1984)
31. Lieb, E. H.: A refinement of Simon's correlation inequality. Commun. Math. Phys. **77**, 127 (1980)

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