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On the Behavior of the Surface Tension for Spin Systems in a Correlated Porous Medium

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We investigate the behavior of various spin-systems that are subject to the highly correlated and extremely diluted quenched disorder as provided by the fractal aerogel model. For these systems, it is (easily) established that, at all temperatures, the free energy is identical to that of the corresponding uniform system. The surface tension, however, behaves quite differently. Foremost, at any fixed temperature corresponding to the low temperature phase in the uniform system, there is a non-trivial curve in the aerogel phase plane dividing high-temperature behavior (zero surface tension) from low-temperature behavior (positive surface tension). The fractal aerogel has two distinctive phases in its own right: gel and sol. In the gel phase, the spin system has zero surface tension at all temperatures. In one region of the sol phase, the surface tension is shown to be equal to its value in the uniform system. Since part of this region borders on the gel phase, a certain portion of the sol/gel phase boundary is also the dividing line between high- and low-temperature behavior. Evidently, in this case, the surface tension is discontinuous at the phase boundary. on the other hand, there are well-defined length scales that diverge as the phase boundary is approached. This demonstrates an absence of scaling in these systems.

KEY WORDS: Aerogels; fractal percolation; surface tension.

INTRODUCTION AND STATEMENT OF RESULTS

Introductory Remarks

In this paper we investigate the properties of spin systems subject to long-range correlated,.quenched dilution. We are motivated in part by a number

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of interesting experiments (e.g., refs. 3 and 10) on the superfluid transition for ^{4}He absorbed in aerogels. Aerogels are very high porosity materials with a wide distribution of pore sizes. Despite the fact that the aerogel displaces only a few percent of the volume of the ^{4}He, it has dramatic effects on the superfluid transition, including an increase in the superfluid density exponent. Several years ago one of us proposed that these experiments could be explained by the effect of a wide distribution of pore sizes (11). The experimental system is modeled by spins on a lattice with random dilution representing the aerogel. The model's geometry is a generalization of Mandelbrot's fractal percolation process (12, 5) and has been referred to as a Mandelbrot aerogel (6). The dilution is constructed in such a way as to create a statistically self-similar distribution of connected pores. The geometry of this model is itself fascinating and apparently displays a nonuniversal percolation transition. Properties of the phase diagram and correlation length for the connectivity of this model were established in a previous paper (6) and are reviewed below. The behavior of spins in this geometry was studied in ref. 11 using renormalization group arguments.

Although the problem of superfluidity in porous media is still controversial, the statistical mechanics of spins models subject to correlated dilution is a fascinating topic in its own right. A number of unexpected phenomena occur which have no analogs in spin systems with short-range correlated randomness. For example, for some parameter values, it seems that there are two distinct transition temperatures, one for thermodynamic properties and a second, lower temperature for properties such as the surface tension, spin stiffness, or superfluid density that require connectivity across the system. In the Mandelbrot aerogel model, the diluted sites constitute a random fractal and approach zero density. Thus, it is intuitively plausible (and proved here) that the thermodynamics of the system is the same as that of the pure system. Nonetheless, for some parameter values, the dilution is capable of disconnecting the pore space. If the pore space is "just barely" connected, the surface tension may vanish at a temperature less than the bulk critical temperature. The way in which the surface tension vanishes is also peculiar. It appears that for some values of the parameters, the stiffness achieves the pure system value until the transition temperature is reached at which point there is a discontinuous drop to zero. Finally, if the dilution is so sparse that it is incapable of disconnecting the pore space there, the stiffness is identical to the pure system value for all temperatures. (This last fact is the main result proved in this paper.) For certain versions of the model, it is also possible to define a correlation length, different from the bulk correlation length, which diverges even when the surface tension is discontinuous. Thus, at least in certain cases, these systems exhibit a manifest absence of scaling.

Weinrib and Halperin (13) discussed a spin model with quenched disorder in which there are long-range correlations in the coupling strengths. The couplings are chosen as Gaussian random variables with correlations decaying as a power of the distance between the bonds. Although one might guess that sufficiently near the critical point, the form of the disorder would become irrelevant so long as the exponents describing the decay of correlations coincide, in fact this is not the case. The effects of fractal dilution are qualitatively quite different from the effects of long, range correlated Gaussian randomness. Indeed none of the facts proved in this paper occur in the Gaussian model; evidently certain notions of universality do not apply when the disorder has long-range correlations.

Description of the Model and Summary of Results

The model that we study was introduced in ref. 11, the starting point of which is a multiscale model of an aerogel (sometimes known as a Mandelbrot aerogel) of the type that was analyzed in ref. 6. Let us first indulge in a brief description of the aerogel proper: The aerogel is defined, for two dimensions, on the unit square. The square is subdivided into N^2 smaller squares which are, independently, "retained" with probability Q or declared to be "pores" with probability $1-Q$. The pores remain as such; however all of the retained squares are further subdivided, again into N^2 pieces (now of linear scale N^{-2}), each of which, independently, may be declared a pore with probability $1-Q$ or be retained for another round of subdivision. After $n-1$ iterations of this procedure, a final step is performed, but this time the retention probability is given by some number p which *a priori* bears no relation to Q. Those cells (now of linear dimension N^{-n}) that have survived are to be thought of as particles, e.g., of silicate, and everything else is pore space. An analogous procedure can be used to construct an aerogel in any spatial dimension.

The above-described aerogels may serve as a disordered medium on which we can define a statistical mechanics problem. The next step is to place spin variables on all the cells of the pores; the spins interact with the usual sort of Hamiltonian for a diluted medium. The choice of Hamiltonian of course depends on the physical system that one wishes to study. (Thus, to make contact with the liquid helium experiments, one should investigate the three-dimensional aerogels with XY spins in the pores.) However, in this preliminary investigation, we will confine attention to the simplest version of the problem, namely the q-state Potts models—and bond percolation—in the two-dimensional aerogel. These systems are interesting enough in their own right and, when the analysis is finally completed, it is

our hope that many of the concepts will see a direct application in the more sophisticated systems

On the basis of the above description alone, it is clear that the bulk thermodynamics is hardly affected by the aerogel disorder. Indeed, on a heuristic level, the gel sites occupy a set to which one can (loosely) assign a Hausdorff dimension of $d_H = 2 - |\log Q/\log N|$; hence the overall clout of the disorder is confined to a vanishingly small faction of the system The result, which we prove later, is that the free energy per spin is identical to that of the uniform system. However, the behavior of a system is not completely determined by its bulk thermodynamics, there is also the arena of surface physics. Thus we turn to the question of surface tension for spin systems in aerogels.

In the analysis of the aerogel model (11, 6), two distinctive phases were discussed, a sol phase and a gel phase. The former was defined by the property that the chances of having a connected crossing between opposite sides of the square in an nth-stage aerogel tended to zero with n. In the latter, these probabilities are uniformly positive. Hence, despite the low density, in this phase, there are gel paths that can fractionate the system.[3]

Let us pause for a brief discussion of the sol–gel phase diagram. (Please consult Fig. 1A.) In the upper left corner—p and Q close to one—lies the gel phase. All else is regarded as sol. However, there are (at least) two distinctive types of sol, which, in a crude manner can be classified as follows: (a) The multiscale parameter Q is too small to allow for gel formation; (b) Q "would have been large enough" but the gel was broken up, at the smallest scale, due to a small value of the parameter p. The division point Q_c is the translation point in the Mandelbrot percolation process (11, 5) (this process may be defined by setting $p = Q$) and the upper portion of the line $Q = Q_c$ constitutes the left boundary of the gel phase.

Consider now the influence, or lack thereof, that is exerted on the spin system by this disorder. In the gel phase, the pores can be isolated from one another and it is plausible that this will cause a depression of the surface tension relative to its value in the uniform system. In fact, a careful look at any reasonable definition of surface tension points to the fact that in this phase, for all positive temperatures, the surface tension will vanish.

[3] It turns out that the notion of gel-connectedness used in ref. 6 is not quite the relevant concept for (nearest neighbor) spin systems defined in the complement of an aerogel. Nevertheless, all the relevant results in ref. 6 go through with an appropriately redefined version of connectedness.

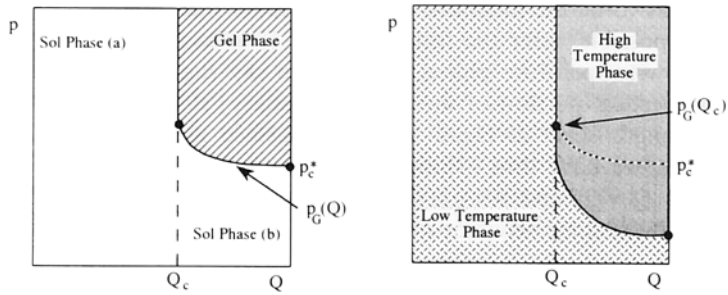

Fig. 1. (A) Sol phases (a) and (b) determined by value of the long-range parameter. p_c^*: *-connected percolation threshold (relevant notion of connectivity). $p_G(Q)$: Lower portion of the sol–gel phase boundary. (B) Vertical portion of the phase boundary coincides with that of the sol–gel boundary. Break occurs below $p_G(Q_c)$ Right endpoint of the phase boundary is presumably the transition point for the single-scale disordered system.

This constitutes our first result. (For one particular definition of the surface tension, this is essentially trivial.)

By contrast, if $Q \leqslant Q_c$ (and, say, $p = 1$) after a finite number of iterations, a connected path of vacant blocks will cross the unit square. As we continue the iteration process, the width of this gap—as measured in lattice spacings—grows. Thus we have a swath of uniform spin system that is of "near thermodynamic proportions" running across the unit square. Under these circumstances, it is clear that if the pure system is in the low-temperature phase, the aerogel spin system will have a positive surface tension. This is indeed the case. Furthermore, and somewhat surprisingly, this surface tension is exactly equal to that of the uniform system. A proof of the latter statement is the major achievement of this paper (Whether or not the result is obvious, a rigorous demonstration is somewhat intricate.)

Let us now consider Fig. 1B, the full phase diagram for the aerogel spin system at some fixed temperature below the transition temperature of the pure system, with p and Q allowed to vary. First, recall that for all p and Q, the system is *thermodynamically* in the low-temperature phase. Nevertheless, as we have just argued, the surface tension exhibits both high- and low-temperature behavior. The boundary between these phases follows the vertical sol–gel phase boundary for large p and then extends into the sol phase. The details of this curve will depend on the temperature, but the general features cannot differ too much from those displayed in the figure.

An eminent property, which may be of some significance, is the fact that at least along part of the phase boundary, the surface tension jumps discontinuously. This effect is reminiscent of an infinite order transition and

indeed seems to be the hallmark of these random fractal systems. Furthermore it is possible, after the fashion of ref. 6, to associate a length scale with the low-temperature phase. Needless to say (again, with rigor, only along the vertical portion of the phase boundary) it is readily established that this length scale diverges. Since the surface tension itself defines a length scale, we have thus exhibited an absence of scaling. Below the gel phase [i.e., $Q > Q_c$ and $p < p_G(Q)$] we suspect that similar and perhaps even more interesting behavior is possible.

Organization of Results

The organization of this paper is as follows:

In Section 1, we first define, in full detail, the aerogels and their associated statistical mechanical models. Then we define, and prove the existence of, the relevant notions of surface tension in these various systems.

In Section 2, we state and prove most of our principal results. First (and readily established) the vanishing of the surface tension in the gel phase at all temperatures (Theorems 2.2 and 2.4) and the existence of a high-temperature phase for p close to $p_G(Q)$ (Theorems 2.3 and 2.4). Then, for the case of percolation, we show that for $Q < Q_c$ (and at bond density above the percolation threshold) the surface tension is equal to the usual surface tension for percolation (Theorem 2.6 and its corollary). We state, but do not prove, in Theorem 2.7 the analogous result for the q-state Potts models. Finally, we conclude this section with some remarks on scaling.

In Appendix A, we provide a proof that the (bulk) fee energy in these models is always equal to that of the uniform system and hence that thermodynamic quantities are insensitive to any of these considerations.

In Appendix B, we start with some technical results concerning the low-temperature phase of the uniform Potts model and then provide a proof of Theorem 2.7.

1. DEFINITIONS AND PRELIMINARY RESULTS

1.1 Definition of the Model

At the foundation of this work is the multiscale model of percolation, introduced in ref. 12, that we will here refer to as the fractal (or Mandelbrot) percolation system. The model has been described elsewhere in detail (e.g., in ref. 5; see also ref. 1 for an alternative discussion), so we will be concise.

The process is defined on the unit square which will be denoted by A_0. Let $N > 1$ denote an integer and let $Q \in (0, 1)$. We divide A_0 into the N^2 smaller squares $[(i-1)/N] \times [(j-1)/N]$, $0 \leqslant i, j \leqslant N$, each of which is independently discarded with probability $1-Q$ or retained with probability Q. The (closure of the) union of the retained squares is denoted by A_1. We obtain $A_{n+1} \subset A_n$ by performing the analogous procedure on all the surviving squares of A_n. The limiting configuration (which is well defined) is denoted by A_∞: $A_\infty = \bigcap_{n=1}^{\infty} A_n$. We use the notation

$$\theta_n(Q) = \text{Prob}(A_n \text{ contains a connected path of surviving squares from the top to the bottom of } [0, 1]^2) \tag{1.1}$$

In Eq. (1.1), two (surviving) squares are deemed to be connected if they share an edge in common. The phase of the Mandelbrot percolation system is indicated by the quantity

$$\theta_\infty(Q) = \lim_{n \to \infty} \theta_n \tag{1.2}$$

If $\theta_\infty > 0$, the system is said to percolate. It is known that there is a $Q_c \in (0, 1)$ above which there is percolation and below which there is not. Furthermore, it is known that $\theta_\infty(Q_c) > 0$.

Next, we describe the (fractal) aerogels that were introduced in ref. 11 and studied in ref. 6. These depend on three parameters, p, Q and N with Q and N exactly as described above and $p \in (0, 1)$. Succinctly put, the aerogels are defined by starting with an (A_n) from the Mandelbrot process. Then, for each A_n, we create a $C_{n+1} \subset A_n$ by subdividing the cells of A_n as before, but this time using p rather than Q for the retention parameter. If $p \geqslant Q$, this is seen to be essentially equivalent to the old fractal percolation problem; however, if $p < Q$, subtle differences may arise. We may denote a generic sequence of aerogels by $((A_0, C_1), (A_1, C_2), (A_2, C_3), \ldots)$, but there is *a priori* no limiting set for the C's. To define our phases in the aerogel problem, we could start as in Eq. (1.1) with A_n replaced by C_n. (This definition was used in ref. 6.) However, for present purposes, the relevant definition is

$$\Theta_n(p, Q) = \text{Prob}(C_n \text{ contains a } *\text{-connected path of surviving squares from the top to the bottom of } [0, 1]^2) \tag{1.3}$$

where, in Eq. (1.3), two squares are considered to be $*$-connected if they contain even a single point in common. The reason for the use of $*$-connectedness in this work will become clear once we introduce the spin

degrees of freedom for these problems. There is no proof of a large-n limit for the $\Theta_n(p, Q)$ (for either definition of connectedness). Hence we define

$$\Theta_\infty(p, Q) = \liminf_{n \to \infty} \Theta_n(p, Q) \tag{1.4}$$

The gel phase consists of those points (p, Q) for which $\Theta_\infty > 0$ and all else is considered to be the sol phase. It is known (6) that the gel phase includes the phase boundary. The relevant features of the phase diagram and some additional notation can be found in Fig. 1a.

We now turn to the definitions for the spin systems that use these aerogels as a background medium. Let $\Lambda_n \subset \mathbb{Z}^2$ denote the sites with x_1 and x_2 coordinates lying between 1 and N^n (so that $|\Lambda_n| = N^{2n}$). We may identify $[0, 1]^2$ with Λ_n in a natural fashion by cutting $[0, 1]^2$ into N^{2n} smaller (disjoint) squares and placing the sites of Λ_n in the centers. In the same way, we may associate C_n with a subset of Λ_n, and of equal notational importance is the complementary set $\Lambda_n \backslash C_n$, which we denote by K_n. The sites C_n are, of course, the gel particles and the K_n represent the pores in which reside the spin degrees of freedom. In the bulk of this paper, we will be concerned exclusively with the q-state Potts models (and, more often than not, Bernoulli percolation), so we defer the formal definitions for general spin systems to Appendix A and focus here on the random cluster representations of these specialized models.

Along with the sites of K_n, we have to consider the sites of the boundary $\partial\Lambda_n$ of Λ_n; as usual, $\partial\Lambda_n$ are those sites in $\mathbb{Z}^2$ that are not in Λ_n but have a neighbor in Λ_n. Between each pair of neighboring sites of $K_n \cup \partial\Lambda_n$, we may envision a bond. The totality of all such bonds will be denoted by $\mathbb{K}_n$. A configuration of bonds is a collection of some of the bonds of $\mathbb{K}_n$ and, generically, will be denoted by an ω. The random cluster measures, for a fixed realization K_n, assign the weights

$$W(\omega) = (1 - e^{-\beta})^{|\omega|} e^{-\beta|\mathbb{K}_n \backslash \omega|} q^{c(\omega)} \tag{1.5}$$

where in the above, β is the inverse temperature, $|-|$ denotes the number of bonds, and $c(\omega)$ is the number of distinct bond clusters—including isolated sites—in the configuration ω. Still, $c(\omega)$ is ambiguous because we have not yet specified the boundary conditions. One alternative is to forbid the occurrence of any bond connecting a site of K_n to a site of $\partial\Lambda_n$ and then to just count the number of components. These are the so-called fee boundary conditions and correspond to free boundary conditions in the spin system. Another possibility is to count all bonds that are connected to any boundary site as part of one single cluster. These are the so-called wired conditions and correspond to the spin system with all the boundary

spins locked in the same state. The sum of all the weights is equal to the partition function for the spin system with the appropriate boundary conditions. We denote these objects by $\mathscr{Z}^f_\beta(K_n)$ and $\mathscr{Z}^w_\beta(K_n)$ respectively.

A little reflection on the weights in Eq. (1.5) (or the spin systems from which they came) reveals that $*$-connectedness is indeed the relevant concept for the gel sites in these systems. Let $V_1 \subset K_n$, $V_2 = K_n \setminus V_1$ and suppose that C_n separates V_1 and V_2. (By this we mean the usual sense of separation: each connected path between V_1 and V_2 that lies in Λ_n uses at least one site of C_n.) Then, by definition, no bonds in $\mathbb{K}_n$ can reach from V_1 to V_2 and, consequently, the interaction term $q^{|c(\omega)|}$ in Eq. (1.5) factors. Under these circumstances, the bond configurations in V_1 and V_2 are statistically independent and C_n has destroyed the coherence between the two halves of K_n. Evidently, then, the relevant sort of connectivity for gel sites is that of the boundaries of connected sets of $\mathbb{Z}^2$, namely $*$-connectedness.[4] In particular, if gel sites $*$-percolate, in accord with the usual infinite-volume notions of percolation, it is obvious that the spin systems can only exhibit high-temperature behavior. However, gel formation in the Mandelbrot aerogels is a subtler form of $*$-percolation and hence a more detailed analysis is required.

The weights in Eq. (1.5) can, of course, be written in terms of weights for *dual* bonds. To this end, consider the sites on the dual lattice $(\mathbb{Z}+\frac{1}{2})\times(\mathbb{Z}+\frac{1}{2})$ which are endpoints of the bonds transverse to the bonds of $\mathbb{K}_n$. Let K_n^* denote the set of these bonds. (The particulars of $\mathbb{K}_n^*$ will depend on the boundary conditions.) For each $\omega \subset \mathbb{K}_n$, we may associate a dual configuration $\omega^* \subset \mathbb{K}_n^*$ by declaring each dual bond $b^* \in \mathbb{K}_n^*$ to be occupied, or open, if the corresponding $b \in \omega$ is vacant. Defining, for future simplicity, $\lambda = e^{-\beta}$, we will simply write

$$W^*(\omega^*) = \lambda^{|\omega^*|}(1-\lambda)^{|\mathbb{K}_n^* \setminus \omega^*|}\, q^{c(\omega)} \tag{1.5*}$$

without bothering to perform a transformation on the term $c(\omega)$.

The essential properties of the measures induced by the weights in Eq. (1.5*) are well known: for $q \geqslant 1$, the measures are FKG and dominate, in the sense of FKG, the (dual bond) Bernoulli measures at (dual) bond density λ.

Of crucial importance in the analysis of these systems is the dual notion of connectivity in Λ_n against the background of disorder sites C_n. As we have discussed, the sites—and hence the bond clusters—of λ_n can be isolated by the presence of $*$-connected circuits in C_n. Thus the gel sites

[4] The reader will recall that two sites on $\mathbb{Z}^2$ are $*$-connected if they are nearest or next nearest neighbors and that if $V \subset Z^2$ is connected with $|V| < \infty$, then the set of sites in the infinite component of $\mathbb{Z}^2 \setminus V$ that have a neighbor in V—e.g., the external boundary—is $*$-connected.

and the dual bonds are allies. The partnership can best be exhibited by sticking with the full lattice λ_n and for the gel configurations $C_n \subset \Lambda_n$ declaring that the quartet of dual bonds surrounding each site of C_n are already serving as open dual bonds. For any C_n and any $\omega \subset \mathbb{K}_n$, along with a specified boundary condition on $\partial\Lambda_n$, it is clear that this leads to the correct counting of $c(\omega)$. Henceforth, when we speak of dual connections, on Λ_n we will always mean connections via occupied dual bonds and the "preoccupied" dual bonds surrounding the gel sites of C_n.

1.2. Surface Tension in a Porous Medium

Our definition of surface tension in these random systems begins in the standard fashion. For example, in the Ising version, for a fixed disorder realization, we consider the ratio of the partition function with plus/minus spins placed on the top/bottom half of $\partial\Lambda_n$ to the partition function in which Λ_n is surrounded by plus spins. In the language of the random cluster representation, it turns out that this is just the probability (for fixed realization C_n) that, in the "wired" ensemble, there is no path of open bonds in K_n connecting the top and bottom halves of $\partial\Lambda_n$. Similar observations hold for all integers $q > 2$. As is traditional, this probability serves to replace the above described ratio in the percolation (or noninteger q) versions of this problem.

For the uniform systems it is well known, at least for $q \geqslant 1$, that this probability (or ratio of partition functions) has the asymptotic scaling of $\exp\{-\sigma(\beta) N^n\}$, where σ (which also depends on q) is called the surface tension. In the random systems, we may denote this quantity by $\exp\{-S_n N^n\}$, but here, $S_n(C_n)$ is itself a random variable. The question before us now is how to go about extracting a limiting value from the sequence (S_n). There are, in fact, two standard procedures for extracting a limit but both of these turn out to be ill suited to our purposes.

The first method is to look for "typical" values of the surface tension and amounts to the (or a) large-n limit of the average of S_n. For systems where the disorder has finite-range correlations, it is expected that the random surface tension will be self-averaging and indeed the existence (with probability one) of a limit for S_n that (with probability one) has the same value for all disorder configurations can be established in a number of cases. Under these circumstances, it is this surface tension that is most likely to be of experimental relevance. Unfortunately for us, this is clearly the wrong object. Consider the gel phase: In this phase we anticipate that the surface tension will vanish. However, it is seen that S_n is equal to the uniform value $\sigma(\beta)$, with probability greater than $(1-Q)^{N^2}$. Hence, self-averaging is destined to fail and, furthermore, if we take the limit of the

average or the average of the limit (assuming that either exists), we end up with the wrong answer.

The second standard method for producing a surface tension is to take seriously the analogy between $e^{-S_n N^n}$ and a correlation function. In this case, the appropriate action would be to average the quantities $e^{-S_n N^n}$ and then take $-N^{-n}$ times the log of this average. The difficulty here is the potential for dominance of the final result by atypical configurations. (And this is a potential source of difficulty in any disordered system.) Consider, for example, the region where $QN^2 < 1$. Here, it is easily shown (5) that with probability one the system is uniform. Hence, in this region, the surface tension should clearly be equal to $\sigma(\beta)$, which, we note, tends to infinity with β. On the other hand, it is easily seen that with probability $p^{N^n} Q^{N^{n-1}} \cdots Q^N$, the surface tension is zero. Evidently, at least for large β, this method also produces the wrong answer.

For these systems, we propose two methods of defining the surface tension. Both definitions have the decisive advantage that they avoid all of the problems discussed above.

In our first definition, we take $R \geqslant 1$ and consider the quantity $e^{(1/R) S_n N^n}$. We define

$$\alpha_R = \lim_{n \to \infty} -\frac{R}{N^n} \log \mathbf{E}(e^{-(1/R) S_n N^n}) \tag{1.6a}$$

and

$$\alpha = \lim_{R \to \infty} \alpha_R \tag{1.6b}$$

In a short-range system that enjoys the "self-averaging" property, one would anticipate that α is equal to the typical surface tension. Notice,that if a significant fraction of realizations of the system have vanishing surface tension, then α vanishes; thus we anticipate that $\alpha = 0$ in the gel phase. Furthermore, the difficulty in the sol phase posed by exponentially rare realizations of the disorder with no surface tension is avoided by the large-R limit.

The second definition is motivated by experimental considerations. In real "fractal" materials such as aerogels there is an upper cutoff beyond which the material becomes uniform. If this length is much less than the macroscopic length scale, it follows that the experimentally accessible surface tension will be self-averaging. On the basis of these considerations, we examine a sequence of models, each of which is actually well defined at both the microscopic and thermodynamic level. The present model, in a vague sense, is the limiting version of this sequence. To define the kth

model, we consider the square lattice and stipulate that each block of side N^k houses an independent copy of C_k. (Thus, in a square of side N^n, it is as though the first $n-k$ iterations of the process somehow resulted in full retention.) Such a system, on $\mathbb{Z}^2$, has correlations that ultimately must be regarded as short-ranged. It is therefore not difficult to believe—and in our case straightforward to prove—that a well-defined typical surface tension exists for these systems. Furthermore, it is clear that these objects are monotone increasing, so, denoting the various surface tensions by $\alpha^*_{[k]}$ we may define

$$\alpha^* = \lim_{k \to \infty} \alpha^*_{[k]} \tag{1.7}$$

We have not succeeded in finding a general relationship between the surface tensions α and α^*. However, it turns out that in all the regions treated in this paper, we can prove that $\alpha = \alpha^*$.

Let us now turn to the elementary task of establishing the existence of these limits.

Proposition 1.1. For the q-state Potts ferromagnets defined against the background of the above-described random aerogels, let $\mathscr{Z}^{\pm}_{\beta}(K_n)$ denote the random partition function that has all the boundary spins above the midline in one particular state and all spins below the midline in another. Let $e^{-S_n N^n}$ denote the ratio $\mathscr{Z}^{\pm}_{\beta}(K_n)/(1/q)\,\mathscr{Z}^{w}_{\beta}(K_n)$. For the general random cluster models with $q \geqslant 1$ let $e^{-S_n N^n}$ denote the probability, in the wired ensemble of K_n, that there is no path of open bonds that connects the top and bottom halves of $\partial \Lambda_n$. Then, for any R,

$$\alpha_R = \lim_{n \to \infty} -\frac{R}{N^n} \log \mathbf{E}(e^{-(1/R)\, S_n N^n})$$

exists and furthermore,

$$\alpha = \lim_{R \to \infty} \alpha_R$$

exists. Next, we consider the q-state Potts ferromagnets or random cluster models with $q \geqslant 1$ in which each square block of $\mathbb{Z}^2$ of the form $n_1 N^k \leqslant x_1 < (n_1+1)\,N^k$, $n_2 N^k \leqslant x_2 < (n_2+1)\,N^k$ is replaces by an independent realization of K_k. Define $\exp(-\mathscr{S}^{[k]}_m m N^k)$ is a similar ratio of partition functions or crossing probabilities for the square of side mN^k bounded by the positive coordinate axes. Then

$$\alpha^*_{[k]} = \lim_{m \to \infty} \mathscr{S}^{[k]}_m$$

exists with probability one and, with probability one, is independent of the configuration. Furthermore,

$$\alpha^* = \lim_{k \to \infty} \alpha^*_{[k]}$$

exists. Finally, as functions of p, Q and β these quantities all exhibit the expected monotonicity properties: α and α^* are both monotone decreasing (i.e., nonincreasing) in p and Q and increasing (i.e., nondecreasing) in β.

Proof. It is noted (without proof) that for integer q larger than one the above specified ratio of partition functions is equal to the probability of observing the above described random cluster events in the associated random cluster model. We will therefore work in the random cluster representation and, to keep things simple, focus exclusively on the percolation case. For any $q > 1$, the arguments are nearly the same provided that some care is taken in dealing with the boundary conditions. Finally we will assume, for simplicity, that N is odd.

In the dual representation, the required event admits the far simpler description that there is a connected dual crossing between the (dual) boundary sites on the midline that are on opposing sides of the boundary. We will therefore work in this representation.

Let us consider first the quantity S_n: A dual crossing of the desired type is achieved if, for example, the N squares across the midline are all retained and N crossings of this type are achieved on the smaller scale squares. This gives the estimate

$$e^{-S_n N^n} \geqslant \chi_{T_1} \cdots \chi_{T_N} e^{-S^{(1)}_{n-1} N^{n-1}} \cdots e^{-S^{(N)}_{n-1} N^{n-1}} \tag{1.8}$$

where χ_{T_m} is the indicator for the event that, on the first iteration, the mth square along the midline is retained and $\exp(-S^{(m)}_{n-1} N^{n-1})$ is the probability of observing a left–right crossing of the described type on the associated smaller scale square. This gives

$$\mathbf{E} e^{-(1/R) S_n N^n} \geqslant Q^N [\mathbf{E}(e^{-(1/R) S_{n-1} N^{n-1}})]^N \tag{1.9}$$

since the quantities $\exp(-S^{(m)}_{n-1} N^{n-1})$ are independent and equal, in distribution, to $\exp(-S_{n-1} N^{n-1})$. Equation (1.9) easily leads to a restricted subadditive bound but it is not, in and of itself, quite enough to ensure the existence of a limit. However, by repeating this sort of argument on k scales we obtain the estimate

$$[\mathbf{E} e^{-(1/R) S_n N^n}] \geqslant Q^{N^k} Q^{N^{k-1}} \cdots Q^N [\mathbf{E} e^{-(1/R) S_{n-k} N^{n-k}}]^{N^k} \tag{1.10}$$

Fixing $n-k=j$, we may rewrite Eq. (1.10):

$$[\mathbf{E}e^{-(1/R)S_nN^n}] \geqslant Q^{[N/(N-1)]N^{n-j}}\mathbf{E}e^{-(1/R)S_jN^j}]^{N^{n-j}} \tag{1.11a}$$

or

$$\frac{-\log \mathbf{e}(e^{-(1/R)S_nN^n})}{N^n} \leqslant \frac{N}{N-1}\frac{1}{N^j}|\log Q| - \frac{\log \mathbf{E}(e^{-(1/R)S_jN^j})}{N^j} \tag{1.11b}$$

From Eqs. (1.11), the existence of the limit α_R is readily established. An elementary convexity argument can be used to show that the α_R are monotone increasing in R and thus $\alpha = \lim_{R\to\infty}\alpha_R$, exists

Turning attention to the quantities $\mathscr{S}_m^{[k]}$, a stronger version of Eq. (1.8) is readily derived; here we have for any positive integers $m_1, m_2, \ldots$ with $\sum_j m_j = m$

$$\exp(-\mathscr{S}_m^{[k]}mN^k) \geqslant \prod_j \exp(-\mathscr{S}_{m_j}^{(j)[k]}m_jN^k) \tag{1.12}$$

where the $\mathscr{S}_{m_j}^{(j)[k]}$ are independent of one another and equal, in distribution, to $\mathscr{S}_{m_j}^{[k]}$. Equation (1.12) easily leads to a subadditive estimate and (the elementary version of) the Kingman theorem (9) may be applied. The result is that, with probability one, the $\mathscr{S}_m^{[k]}$ converge to some $\alpha_{[k]}^*$ which, with probability one, is a constant.

It is clear that the $\mathscr{S}_{Nm}^{[k]}$ are distributionally smaller than the $\mathscr{S}_m^{[k+1]}$ since the configurations associated with the latter can be constructed out of those associated with the former by the insertion of an additional (large) scale of random pores. Hence the $\alpha_{[k]}^*$ are increasing and the large-k limit, α^* exists.

Finally, the resulting quantities α and α^* inherit the various described monotonicities that are manifest in the sequences of which they are the limits. ∎

For fixed, sufficiently large β, let us examine the phase diagram in the p, Q unit square As usual, we may define the high-temperature phase (e.g., according to α) as the set of points $(p, Q) \in [0, 1]^2$ such that $\alpha(p, Q) = 0$. We claim that due to the monotone properties of $\alpha(p, Q)$, there is a well-defined phase boundary: Indeed, this phase boundary is just the graph of the function $p_H(Q) = \sup_p\{\alpha(p, Q) > 0\}$ along with (at most) a countable number of vertical segments for Q in some interval (which turns out to be $(Q_c, 1]$).

We conclude this section with the following observation:

Remark. In disordered systems with short-ranged correlations, it is generally believed that quantities like α_R (e.g., inverse correlation lengths)

are of the same order of magnitude. In particular—even for the aerogels—it is easy to show that if α_R vanishes for any R, then all of the α_R are zero. Indeed, suppose that $e^{-S_n N^n}$ satisfies

$$\mathbf{E}(e^{-S_n N^n}) \geqslant e^{-t_n N^n} \tag{1.13}$$

with $\lim_{n\to\infty} t_n = 0$. Then we have

$$\text{Prob}(e^{-S_n N^n} \geqslant e^{-2t_n N^n}) \geqslant \frac{e^{-t_n N^n} - e^{-2t_n N^n}}{1 - e^{-2t_n N^n}} \geqslant \frac{1}{2} e^{-t_n N^n} \tag{1.14}$$

But then for any R,

$$\mathbf{E}(e^{(1/R) S_n N^n}) \geqslant \frac{1}{2} e^{-t_n N^n} e^{-(2/R) t_n N^n} \tag{1.15}$$

and hence $\alpha_R \leqslant R \times [\lim_{n\to\infty}(1 + 2/R)\, t_n] \to 0$. It is therefore a little surprising that in this system the α's do not scale together. Indeed, it turns out that as we approach a certain portion of the phase boundary, for any finite R, α_R tends to zero continuously while α remains uniformly positive

2. BEHAVIOR OF THE SURFACE TENSION FOR SYSTEMS IN AEROGELS

2.1. Results for the High-Temperature Phase

Our first substantive results concern the "obvious fact" that in the gel phase, the surface tension is zero. For the quantity $\alpha(p, Q, \beta)$, this is indeed obvious and is an immediate consequence of the following:

Lemma 2.1. Let T_n denote the probability of observing in the wired ensemble of Λ_n, a dual path (consisting of occupied dual bonds and/or occupied gel sites) connecting the left and right sides of Λ_n. Suppose that the average of T_n is bounded away from zero uniformly in n:

$$\mathbf{E}(T_n) \geqslant \kappa > 0$$

Then the surface tension, $\alpha(p, Q, \beta)$, is zero.

Proof. Let us again assume, for simplicity, that N is odd. We claim that the statement $\mathbf{E}(T_n) \geqslant \kappa$ implies that there are (deterministic) sites $a \equiv a(n)$ and $b \equiv b(n)$ on the left and right sides of the square such that the average probability that a and b are connected is bounded below by a power of N^n. Indeed, for any one of the sites $a_1, a_2, \ldots$ on the left side

of A_n and any of the sites $b_1, b_2, \ldots$ on the right, denote by $T_n^{[a_i, b_j]}$ the probability of observing a dual connection between the sites a_i and b_j. (We remark that, as far as the aerogel disorder is concerned, this is still a random variable. Indeed, any of the quantities $T_n^{[a_i, b_j]}$, and even T_n, can —for integer q greater than one—be written as the ratios of partition functions with various boundary conditions.) Clearly, in each disorder configuration,

$$T_n \leqslant \sum_{i,j} T_n^{[a_i, b_j]} \tag{2.1}$$

Hence, there must be one pair $[a, b]$ for which

$$\mathbf{E}(T_n^{[a, b]}) \geqslant \frac{\kappa}{N^2 n} \tag{2.2}$$

Of course it is tempting to say that the privileged sites are just the ones in the middle, but there is no real way of proving this. However, with only a little work, we can use the above to show that $e^{-S_n N^n}$ has subexponential behavior with probability that tends to zero slower than exponentially fast.

Indeed, let us consider the N^{n-k} squares of side N^k which, as in Proposition 1.1, bridge the middle of the square of side N^n. With probability exceeding

$$Q^{N^{n-k}} Q^{N^{n-k-1}} \cdots Q > Q^{N^{n-k}/(N-1)}$$

all of these squares and all of their ancestral squares are retained. If, in each of the squares of scale N^k we insist on seeing translations and reflections of dual bond connections between the sites corresponding to $a(k)$ and $b(k)$, then we have produced sites a^* and b^*, within a distance N^k of the midline, such that

$$\mathbf{E}(T_n^{[a^*, b^*]}) \geqslant \left[\frac{\kappa}{N^{2k}}\right]^{(n-k)} Q^{N^{n-k}/(N-1)} \tag{2.3}$$

It is noted that the various events and functions under consideration are all of the same FKG type and that the Harris–FKG inequality has been instrumental in the derivation of Eq. (2.3). Finally, with probability larger than $p^{N^k} Q^{N^k/(N-1)}$, we can ensure that all sites with $x_1 = 1$ or $x_1 = N^n$ that are a distance closer than $\frac{1}{2}N^k$ from the midline are gel sites. Under these circumstances, whenever there is a connection between our favored a^* and

b^*, there is also a connection between the midpoints of the left and right boundaries. Evidently, making further use of the Harris–FKG inequality,

$$\mathbf{E}(e^{-S_n N^n}) \geqslant \left(\left[\frac{\kappa}{N^2 k} \right]^{(n-k)} Q^{N^{n-k}/(N-1)} \right) p^{N^k} Q^{N^k/(N-1)} \tag{2.4}$$

Choosing, e.g., $k = n/2$, Eq. (2.5) easily gives us $\alpha_1 = 0$, from which it follows (cf. the remark at the end of Section 1) that $\alpha = 0$. ∎

As an immediate consequence, we get the following result.

Theorem 2.2. Let (p, Q) be any point in the gel phase. Then, in any of the q-state models with $q \geqslant 1$,

$$\alpha(p, Q, \beta) = 0$$

for all β.

Proof. By definition, in the gel phase, there is a left–right crossing of Λ_n by gel sites with probability uniformly bounded below away from zero. In these configurations, we have T_n equal to one even before we consider the statistical mechanics. ∎

The above is, of course, not particularly surprising: percolation of the aerogel has broken the space into disconnected regions that act independently. Indeed, it is clear that Theorem 2.2 holds for any statistical mechanics model in an aerogel for which a surface tension of the form $\alpha(p, Q, \beta)$ can be defined. [We will deal with the other surface tension, $\alpha^*(p, Q, \beta)$, following Theorem 2.3.] As noted, the result is valid all the way down to zero temperature. It is therefore not hard to believe that just below the sol–gel phase boundary $[p \lesssim p_G(Q)]$ similar results should hold at finite temperatures and in particular at temperatures well below the transition temperature of the uniform system. For our spin systems, we will make (rather inefficient) use of the finite temperature to enhance the (effective) gel-site population and drive the system back into the gel phase.

Recall from our discussion of duality that the dual measure has the bond density parameter $\lambda = e^{-\beta}$ and, for $q > 1$, these measures dominate, in the sense of FKG, the ordinary bond percolation problems at density λ. (A proof of this can be found in ref. 8.) This dominance will allow us to establish the above assertion using independent percolation methods.

Theorem 2.3. Let $\lambda = e^{-\beta} > 0$ and $Q > Q_c$. Then there is a $p(\lambda) < p_G(Q)$ such that for all $p > p(\lambda)$, $\alpha(p, Q, \beta)$ vanishes. In particular

(but of significance only if β is greater than the transition temperature) we may estimate

$$p(\lambda) \leqslant \frac{p_G(Q) - [1-(1-\lambda)^{1/2}]^4}{1-[1-(1-\lambda)^{1/2}]^4} \equiv \tilde{p}(\lambda)$$

Proof. It is sufficient, by the dominance argument, to establish this in the case $q = 1$. In our discussion of duality it was argued that a single gel site is "as good as" having the four dual bonds that surround it in an open state. Conversely, we might say that if the four dual bonds surrounding a given site are open, this is "as good as" having a gel site present. We cannot, however, just write down a factor of λ^4, because these effective site variables are not independent. But we can circumvent this little problem with the following device: Let us imagine that each (dual) bond is replaced by two possible bonds, a red one and a yellow one, that operate independently at densities $y(\lambda)$ and $r(\lambda)$, respectively. We will agree to call the original bond open if either the yellow bond or the red bond is open. To fix the density, we require that $r + y - yr = \lambda$ or, if $y = r$, $y = 1-(1-\lambda)^{1/2}$. Now, we may claim an effective site event for sites on the odd sublattice if they are surrounded by four yellow bonds and on the even sublattice if they are surrounded by four reds. This gives us an overall "density" of $p + (1-p)[1-(1-\lambda)^{1/2}]^4 = \tilde{p}(\lambda)$.

Using the above, it is easily seen that

$$\mathbf{E}(T_n) \geqslant \Theta_n(\tilde{p}(\lambda)), Q) \tag{2.5}$$

which, if $p(\lambda)$ matches or exceeds $p_G(Q)$, we agree is uniformly positive. The result now follows from Lemma 2.1. ∎

We will dispense with "the other surface tension" momentarily; our first step will be the development of some (inevitable) notation for describing the systems with only k scales of disorder.

Definition. Consider the usual Mandelbrot percolation process down to the nth level and let us discuss an equivalent construction of the set A_n that will allow us some additional flexibility. The new construction is implemented by working outward from the smallest scales: We start with the unit square cut into N^{2n} smaller squares (or, if easier to visualize, the lattice Λ_n). Let G_n denote the random subset that is obtained by independently deleting [or retaining] each square of side N^{-n} with probability $1-Q$ [or Q]. In other words, so far, we have independent percolation at density Q on a lattice of size $N^n \times N^n$. Regardless of what transpired in the formation of G_n, let G_{n-1} be constructed in a similar fashion, but this time we will identify each $N^{-(n-1)} \times N^{-(n-1)}$ square as a single site to be

retained or discarded. (Or, back on Λ_n, each $N \times N$ block of sites is independently retained or discarded.) In a similar manner, we can obtain any G_{n-k} with $0 \leqslant k < n$. We let

$$G_n^{(k)} = \bigcap_{j=0}^{k-1} G_{n-j} \tag{2.6}$$

With the obvious identifications, it is clear that $G_n^{(n)}$ is statistically equivalent to the set A_n.

The above construction can also be implemented for the Mandelbrot aerogels: Here the first step takes place using p as the retention parameter and the final $(k-1)$ steps are the same as above. In these cases, we will use the notation $H_n^{(j)}$ (as opposed to a G) to underscore the distinction between the aerogel-type or the unadorned Mandelbrot-type sets

The sets $H_n^{(k)}$ and $G_n^{(k)}$ are exactly the k-scale systems restricted to a lattice of scale N^n. Thus, the $H_n^{(k)}$—with n large—will be used in our discussion of the $\alpha^*_{[k]}$. However, these objects are also essential for our analysis of the full process: $G_n^{(k)}$ may be used for probing the system down to the scale N^{-n} without having exploited the benefit of the larger-scale vacancies. These, in turn, can be kept in reserve as an independent operating force. In any case, if we allow $n \to \infty$ with k fixed (and large), we have at our disposal a genuine thermodynamic system with many features of the current idealization.

As a technical device, we will also consider configurations $G_{n;\ell}^{(k)}$ on $\Lambda_{n+\ell}$ that are formally equivalent to the configurations $G_n^{(k)}$ except that the individual vacant/occupied sites in $G_n^{(k)}$ are now identified as $N^\ell \times N^\ell$ blocks of sites with the appropriate character. Put differently, $G_{n;\ell}^{(k)}$ is constructed from $G_n^{(k)}$ by performing an additional ℓ subdivisions under the agreement that in these subdivisions, full retention is achieved. Back on the unit square, it is clear that any events concerning the $G_{n;\ell}^{(k)}$ can be measured at the level of the $G_n^{(k)}$; in certain cases we will not even make any notational distinctions.

We are now ready for our discussion of the k-scale surface tensions in the high-temperature regimes.

Theorem 2.4. Under the explicit condition of Theorem 2.3, namely

$$p \geqslant \frac{p_G(Q) - [1-(1-\lambda)^{1/2}]^4}{1-[1-(1-\lambda)^{1/2}]^1} \qquad \text{and} \qquad Q \geqslant Q_c$$

(which includes but is not limited to the entire gel phase) the surface tension $\alpha^*(p, Q, \beta)$ is zero.

Proof. By the arguments of Theorem 2.3, the lower bound on p effectively allows us to devote all of our attention to the aerogel model at $p \geqslant p_G(Q)$. Let us therefore examine these systems on $\mathbb{Z}^2$ with only k scales of disorder.

Consider a $2N^n \times N^n$ lattice housing two independent disorder realizations $H_n^{(k)}$ and $H_n^{(k)'}$ alongside one another. Let $\Theta_{2,n}^{[k]}(p, Q)$ denote the probability that the configurations $(H_n^{(k)}, H_n^{(k)'})$ have a left–right $*$-connected crossing of this 2×1 rectangle composed of occupied gel sites. We claim that if $Q \geqslant Q_c$ and $p \geqslant p_G(Q)$, then $\Theta_{2,n}^{[k]}(p, Q) \geqslant 1 - e^{-\gamma_k N^n}$ for some $\gamma_k > 0$.

Indeed, by standard $2d$ rescaling arguments that go back to ref. 1, if for some n_0, $\Theta_{2,n}^{[k]}$ exceeds, e.g. 15/16, then the preceding can be established with $\gamma_k \geqslant \text{const.} N^{-n_0}$. On the other hand, suppose that $\Theta_{2,n}^{[k]}$ gets *very* small. This of course implies that the complementary event, an "easy" way connected crossing of the $2N^n \times N^n$ lattice by vacant squares, gets very close to one. Using the arguments of ref. 4, Lemma 2.6 (or ref. 6, Proposition 2.1), it can be shown that with the help of some larger-scale vacant pores, these short way crossings can be pasted together to achieve long way crossings of yet larger 2×1 rectangles with high probability. (Say, larger than 15/16.) Once this has been achieved, further "old-style" rescalings send the probability of vacant crossings of even larger boxes to one exponentially fast. This in turn implies that $\Theta_n(p, Q) \to 0$, which is not considered sporting if $p \geqslant p_G(Q)$.

Thus, the only remaining alternative to the claim is that the $\Theta_{2,n}^{[k]}(p, Q)$ are uniformly bounded away from 1 and 0. We will show that these circumstances lead to essentially the same scenario as the second case, in particular, they will allow us to find a $\tilde{k} > k$ such that $\Theta_{2,n}^{[\tilde{k}]}(p, Q)$ is again very small.

To this end, let us discuss a relevant notion of disjoint crossings for multiscale percolation. Consider the tilings of the $2N^n \times N^n$ lattice by the $N^k \times N^k$ blocks that represent the largest potential scale of vacancy. We will say that two distinct $*$-connected gel paths are k-scale disjoint if the set of $N^k \times N^k$ blocks visited by these paths are disjoint. We will say that there are m k-scale disjoint paths if there are m distinct paths, each pair of which is k-scale disjoint.

A careful reading of the van den Berg–Kesten inequality (2) lets us know that the probability of more than m k-scale disjoint gel paths connecting the left and right sides of the rectangle is bounded above by $[\Theta_{2,n}^{[k]}(p, Q)]^m$. (This estimate can also be obtained inductively by conditioning to the lowest connected cluster of $N^k \times N^k$ blocks which contain the first $m-1$ k-scale disjoint paths.) Letting $\varepsilon \in \mathbb{R}^+$ and assuming that $\Theta_{2,n}^{[k]}(p, Q)$ is uniformly bounded away from one, then for m large enough,

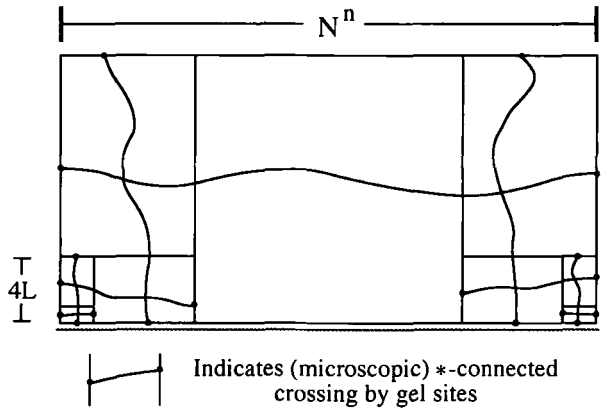

Fig. 2. A construction of $\mathscr{V}_n^{(L)}$.

for all n, there are fewer than m k-scale disjoint gel crossings of the $2N^n \times N^n$ lattice with probability exceeding $1 - \frac{1}{2}\varepsilon$.

Under the condition that there are fewer than m k-scale disjoint crossings, it is clear that there can be no more than mk disjoint crossings at the microscopic level. This means that there is a set of no more than km gel sites which, if removed would destroy the event of a left–right gel crossing. By incorporating the action of scales larger than N^k, we have the power to do just that. Indeed, conditioning to any particular set of fewer than km vulnerable sites (these collections must, of course, be ordered in some fashion so as to form a disjoint partition) and using scales up to $N^{\tilde{k}}$, we find that the crossing is destroyed with probability exceeding $(1 - Q^{\tilde{k}-k})^{km}$.[5] In the preceding, we have tacitly assumed that $n \gg \tilde{k}$ and we have used the Harris–FKG inequality. Thus, for $\tilde{k}$ and n large enough, we have demonstrated [under the assumption that $\Theta_{2,n}^{[k]}(p, Q)$ is uniformly bounded away from 1] that $\Theta_{2,n}^{[\tilde{k}]}(p, Q)$ is small. This, in turn, starts a chain of events, the ultimate conclusion of which is the falsehood that $p < p_G(Q)$. Our claim is now established.

Let $\mathscr{V}_n^{(L)}$ denote the set of configurations $H_n^{(k)}$ in which there is a left–right crossing of Λ_n by gel sites that starts and ends within a distance L of the midline. With the "stated claim" in hand, on the basis of Fig. 2 (and

[5] The authors regard this argument as a model of inefficiency. In principle under the above stated condition, it should be possible to prove that the removal of only m blocks of size as big as $N^k \times N^k$ destroys the event of a crossing. Since the above removal procedure employs blocks of vacancy starting at this scale, one should be able to obtain the improved estimate of $(1 - Q^{\tilde{k}-k})^m$ for the probability of destroying the crossing using $k' - k$ additional scales.

the Harris–FKG inequality) it is not hard to see that for L large and C some fixed finite constant,

$$\text{Prob}_{p,Q}(\mathscr{V}_n^{(L)}) \geqslant 1 - Ce^{-\gamma_k L} \tag{2.7}$$

holds uniformly in n. For any configuration in $\mathscr{V}_n^{(L)}$, it is clear that the surface tension is not larger than $2L\beta/N^n$. In those configurations where the event fails, we may estimate the surface tension by its uniform system value, $\sigma(\beta)$. Thus we arrive at

$$\mathbf{E}(\mathscr{S}_n^{[k]}) \leqslant \frac{2L\beta}{N^n} + Ce^{-\gamma_k L}\sigma(\beta) \tag{2.8}$$

Letting n then L, go to infinity, we get $\alpha_{[k]}^* = 0$ for all k. This is the desired result. ∎

This concludes our analysis of the high-temperature phase. In cases where the surface tension is positive, the analysis requires a more sustained effort.

2.2. Results for the Low-Temperature Phase

In the remainder of this paper, our principal results will concern the region of the phase diagram $Q \leqslant Q_c$, $0 \leqslant p \leqslant 1$. Since we are now trying to establish results that favor the collective action in the pore spaces, it follows that the worst case scenario is when $p = 1$. (This is not nearly as drastic as it seems, even in the presence of an underlying spin system, $p = 1$ is only slightly different from the case $p = Q$.) Thus, the key condition is $Q \leqslant Q_c$ and, as will become apparent, here the physics is determined by pore events on large scales.

The focus of this subsection will be a demonstration that for $Q < Q_c$, even the surface physics is (by and large) unaffected by the disorder. In particular, if the uniform system is in the low-temperature phase, not only are the α's positive, but they are both equal to the surface tension of the uniform system. Ultimately, this result will be established for all the q-state random cluster models with $q \geqslant 1$. However, for $q > 1$, there are spurious complications associated with the distinctive types of boundary conditions. The authors feel that the percolation case ($q = 1$) captures the essential spirit of the argument—and is complicated enough in its own right. Therefore, in this section we will confine attention to this case and deal with the full blown random cluster problem in Appendix B.

In ref. 6, a correlation length N^{k_0}, was defined by looking at the configurations (A_k, A_k') on $[0, 2] \times [0, 1]$ and finding the smallest k, k_0 for

which a left–right connected "crack" of vacant squares appeared with probability exceeding some threshold value. For $Q < Q_c$, the above-described k_0 is finite and, on this basis, it follows rather easily that for the configurations $(G_m^{(k)}, G_m^{(k)'})$ with $k \gtrsim k_0$, such cracks occur with probability at least as large as $1 - \exp\{-\text{const} \cdot N^m/N^{k_0}\}$. These cracks will be instrumental in establishing our principal result of this subsection

Let us briefly outline the motivation and strategy of the forthcoming: Clearly, when a crack separates the two relevant sides of a box, the "surface tension" is positive. Indeed, if a crack appears in the mth iteration and we measure the surface tension on the 2×1 square, the probability of a dual path between the left and the right is less than the probability that there is a path across this crack. If $n = m + \ell$, this in turn is bounded above by the order of $e^{-\sigma N^\ell}$. Thus we can already state, informally, that the surface tension is larger than $(1/N^m)\sigma$ with probability exceeding $1 - \exp\{-\text{const} \cdot N^m\}$. Unfortunately, this probability is not exponentially close to unity with the scale of the system—here N^n—and it is conceivable that smaller surface tensions (occurring in other disorder configurations) could dominate the overall process. Thus, if we keep m fixed, the probabilistic estimate is not sharp enough and if we allow $m \to \infty$ with n, our estimate on the surface tension deteriorates completely.

To circumvent these difficulties, we will use the above described crack events on *small* scales to create a network of cracks. We will refer to these as fissures. (Although these fissures may be considered small relative to the scale of the full system, they will still be appreciable with respect to the scale of the lattice spacing.) The fissures alone will be enough to guarantee the positivity of the surface tension throughout the low-temperature phase and the argument is presented as a separate proposition (Proposition 2.5). Having achieved this, there will be some further discussion in which we will describe how to bring to bear cracks on successive scales. The combination of the crack events on all scales is then shown to drive the surface tension to its uniform-system value; this is the subject of Theorem 2.6.

Proposition 2.5. Consider the percolation ($q = 1$) random cluster aerogel with the bond parameter $1 - e^{-\beta}$, in excess of the (uniform $\mathbb{Z}^2$-bond) percolation threshold and with $Q < Q_c$. Then for any p, $\alpha(p, Q, \beta) > 0$.

Proof. Obviously it is sufficient to discuss only the case $p = 1$. Consider the configurations (A_m, A'_m) on $[0, 2] \times [0, 1]$ and let k_0 denote the smallest integer for which the probability of a connected vacant crossing (crack) between $x = 0$ and $x = 2$ exceeds, e.g., $1 - (25e)^{-1}$. Let $r \gtrsim k_0$ and let m denote a fixed integer that is large compared with r, the precise value

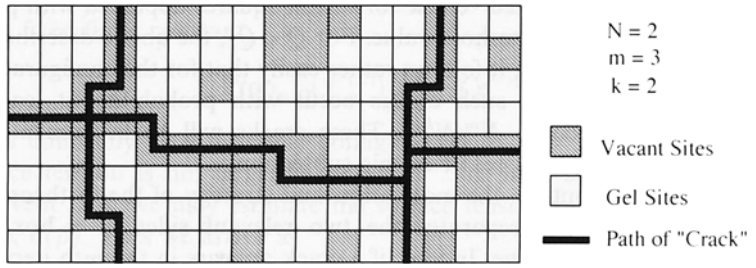

Fig. 3. The event $F_{m,k}$.

of which will be determined later. Let $F_{m,r}$ denote the set of configurations $(G_m^{(r)}, G_m^{(r)'})$ on $[0,2]\times[0,1]$ in which all of the following cracks separating the various sides appear:

(i) a top–bottom connected crossing of vacant squares restricted to the region $[1,2]\times[0,1]$

(ii) a top–bottom connected crossing of vacant squares restricted to the region $[0,1]\times[0,1]$ (2.9)

(iii) a left–right connected crossing of vacant squares

The event $F_{m,r}$ is depicted in Fig. 3. Using the arguments of the type found in ref. 6, (cf. Theorem 3.5 and its proof), it is not hard to show that

$$\text{Prob}(F_{m,r}) \geqslant 1 - c_1 \exp(-c_2 N^{(m-k_0)}) \tag{2.10}$$

with c_1 and c_2 finite constants. Indeed, these are just the usual percolation rescaling lemmas.

Next, in the 2×1 square, let us examine the random cluster configurations after an additional ℓ iterations, performed with full retention, i.e., the configurations $(G_{m,\ell}^{(r)}, G_{m,\ell}^{(r)'})$. For the benefit of the dual random cluster configurations, we will assume that there are free boundary conditions on $[0,2]\times[0,1]$; however, we will also assume that $\sigma\ell$ is large compared with unity. As will become apparent, the latter represents a significant driving force. Let us now see what happens if, during the r iterations, (the analog of) the event $F_{m,r}$ occurs. In particular, under these circumstances, we consider the probability of observing any of the following dual random cluster connections:

(i) a path connecting any dual site on the midline $(x_1 = 1)$ to one on the right $(x_1 = 2)$

(ii) a path connecting any dual site on the midline $(x_1 = 1)$ to one on the left $(x_1 = 0)$ (2.11)

(iii) a path connecting any dual site on the top $(x_2 = 1)$ to one on the bottom $(x_2 = 0)$

An illustration of a similar event defined under slightly different circumstances can be found in Fig. 5, Appendix B.

Given that the event "$F_{m,r}$" has occurred, any of the paths described in Eq. (2.11) must make a run across their counterpart cracks described in Eq. (2.9). Now these cracks are at least as wide across as N^{ℓ} lattice spacings. For percolation, it is easy to show that the probability of any of the above-mentioned paths does not exceed $2(N^{m+\ell})^2 e^{-\sigma N^{\ell}}$, where $\sigma > 0$ is the usual surface tension for Bernoulli percolation (i.e. σ is the inverse of the correlation length for the dual connectivity function).

Let n be large compared with m and let us only consider the disorder configurations $G^{(r)}_{n;\ell}$. We will now estimate the probability of observing a path across $\Lambda_{n+\ell}$ by dual bonds and gel sites. For convenience, we divide $\Lambda_{n+\ell}$ into square pieces of side $N^{m+\ell}$ lattice spacings. To each path, we associate a connected cluster of nonoverlapping 2×1 rectangles (each rectangle consisting of two of the above squares) by the following construction: We say that a rectangle has been *threaded* by the path if the path traverses the rectangle in any of the ways described in Eq. (2.11). Our cluster will be a collection of rectangles threaded by the path. Let us track the path starting from the right side of $\Lambda_{n+\ell}$. Hopefully, the path starts off by threading a rectangle whose long side coincides with the boundary of $\Lambda_{n+\ell}$. In this case, the said rectangle is the first to join the cluster. On the other hand, it may be the case that the path simply plows through the first box it encounters, in which case we will *define* the first rectangle of the cluster to be this box together with the one directly above it. (These two possible scenarios are depicted in Fig. 4A.) Moving along the path, let us say that we have determined the first few rectangles of the cluster. We construct the "boundary" of the existing cluster out of our squares of side $N^{m+\ell}$ using the usual notions of $\mathbb{Z}^2$ connectivity: Here we regard the cluster as a $*$-connected object in $\mathbb{Z}^2$ so that its boundary will end up being connected. (The "sites" of this $\mathbb{Z}^2$ are squares of side $N^{m+\ell}$.) The boundary consists of those squares not in the cluster but sharing an edge or a corner with a square that is. The external boundary is the (edge) connected component that can be reached from infinity without touching the cluster and the boundary is the intersection of this object with the collection of squares that constitute $\Lambda_{n+\ell}$. The next rectangle to join the cluster is, hopefully,

the first 2×1 rectangle in the boundary to be threaded by the path. However, there is again the possibility that the path will cross the boundary in the confines of a single square. In this case we will (artificially) adjoin this square to one of its neighbors in the boundary using some deterministic rule, for example $N > S > E > W$. It is noted that the jth and $(j+1)$th rectangles to join the cluster need not share a side (or even a corner) in common. A few steps of this procedure are illustrated in Fig. 4B. Observe that with each additional rectangle, the "furthest leftward" progress of the cluster is increased by no more than the short length $(N^{m+\ell})$ of the constituent rectangles. We stop the process when the cluster first touches the left side of $\Lambda_{n+\ell}$ and say that such a cluster has *spanned* $\Lambda_{n+\ell}$.

Let $\mathscr{C}$ denote a generic spanning cluster that might have been constructed in the fashion just described. In what is to follow, we do not distinguish geometrically identical clusters, i.e., clusters that were pieced together in a different order. We will consider the equivalence class of all dual paths crossing $\Lambda_{n+\ell}$ giving rise to the cluster $\mathscr{C}$. By slight abuse of notation, we will also use $\mathrm{Prob}(\mathscr{C})$ to denote the probability (in the configuration $G^{(r)}_{n;\ell}$ or $A_{n+\ell}$) of observing any of the paths in the $\mathscr{C}$ equivalence class. It is clear that

$$e^{-S_{n+\ell} N^{n+\ell}} \leqslant \sum_{\mathscr{C}:\, \mathscr{C} \text{ spans } \Lambda_{n+\ell}} \mathrm{Prob}(\mathscr{C}) \tag{2.12}$$

It is further clear that the probability of $\mathscr{C}$ is less than the product of the probabilities of observing the individual threading events in the constituent rectangles. We may now examine each rectangle and determine whether (the appropriate translations and/or rotations of) the event $F_{m,r}$ occurs in the configuration $G^{(r)}_{n;\ell}$. If not, we will estimate the probability from above by unity and otherwise, we obtain a (multiplicative) factor of $2(N^{m+\ell})^2 e^{-\sigma N^{\ell}}$.

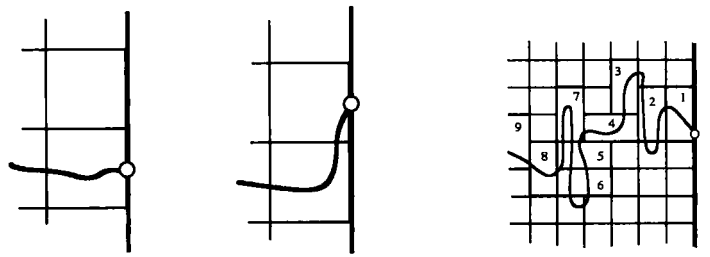

Fig. 4.

Recall that the probability of $F_{m,r}$ is close to unity—at least if m is large; for notational simplicity, let us denote our estimate of this probability by $1-\varepsilon$ (i.e., $\varepsilon = c_1 \exp[c_2 N^{(m-k_0)}]$). Thus, a typical cluster $\mathscr{C}$ that consists of $|\mathscr{C}|$ rectangles has only $\varepsilon\,|\mathscr{C}|$ threading factors that need to be estimated by unity. However, we will end up allowing for a lot more leeway. Let $f \in (\varepsilon, 1)$. In a configuration $G^{(r)}_{n;\ell}$, we will say that a spanning cluster $\mathscr{C}$ is f-normal if a fraction at least as large as $1-f$ of its rectangles experience the analog of the $F_{m,r}$ event. It is noted that the "normalcy" of a given cluster may be computed using binomial statistics since, due to the cutoff after $r+\ell < m+\ell$ scales, the disorder configurations in the individual rectangles are independent.

As a final step, we will say that the configuration $G^{(r)}_{n;\ell}$ is f-normal (or that $\Lambda_{n+\ell}$ is f-normal) if all possible clusters that span $\Lambda_{n+\ell}$ are f-normal. It is easy to estimate

$$\begin{aligned}\text{Prob}(G^{(r)}_{n;\ell} \text{ is not } f\text{-normal}) &\leqslant \sum_{\mathscr{C}} 2^{|\mathscr{C}|} \varepsilon^{f|\mathscr{C}|} \\ &= \sum_{K} \mathscr{N}(K)\, 2^K \varepsilon^{fK} \end{aligned} \tag{2.13}$$

where $\mathscr{N}(K)$ is the number of spanning clusters with exactly K constituent rectangles. The quantity $\mathscr{N}(K)$ can be estimated by the usual sort of polymer arguments: In particular, if we define $\tilde{\mathscr{N}}(K)$ to be the number of distinct clusters of this type modulo translation, it is not hard to show that $[\tilde{\mathscr{N}}(K)]^{1/K}$ tends to a definite limit, e^{ϖ} and that this is also the limit of $[\mathscr{N}(K)]^{1/K}$. Thus one need only verify that ϖ is finite. This can be accomplished in any number of ways. For example, since a neighboring pair of rectangles can only coordinate in 16 possible ways, it follows that $\tilde{\mathscr{N}}(K)$ is less than the number of distinct random walks of length $2K$ and coordination number 16, i.e., $(16)^{2K}$. The minimum value of K is clearly $(N^n/N^m)-1$ and there are also exactly this many "starting positions". Thus if $\varepsilon^f (\equiv c_1 \exp[-f c_2 N^{(m-k_0)}])$ is sufficiently small, $\Lambda_{n+\ell}$ is f-normal with probability exponentially close to one in the (linear) scale, $N^{n+\ell}$.

In the event that $\Lambda_{n+\ell}$ is not f-normal, we will estimate $e^{-S_{n+\ell} N^{n+\ell}}$ by unity; however, if $\Lambda_{n+\ell}$ is f-normal, we have, from Eq. (2.12),

$$(e^{-S_{n+\ell} N^{n+\ell}} \mid \Lambda_{n+\ell} \text{ is } f\text{-normal}) \leqslant \sum_{K \geqslant N^n/N^m} \mathscr{N}(K)[[2N^{2(m+\ell)}]\, e^{-\sigma N^{\ell}}]^{(1-f)K} \tag{2.14}$$

Now to set the constants. First let H be large enough to ensure that $2e^{\varpi} c_1 e^{-c_2 H}$ is less than one; to be definitive, say that (for all K sufficiently

large) $2\mathcal{N}(K)^{1/K} c_1 e^{-c_2 H} \leqslant e^{-1}$. Now find the $m > k_0$ that maximizes $(1 - H/N^{(m-k_0)}) N^{-m}$ and set $f = H/N^{(m-k_0)}$. The quantity ℓ is yet to be determined, but let us keep in mind, for the moment, only the fact that it is finite. We see that in a fraction smaller than e^{-N^n/N^m} of the configurations, $\Lambda_{n+\ell}$ may not be f-normal. In these configurations, we must estimate $e^{-S_{n+\ell} N^{n+\ell}}$ (and hence $e^{-(1/R) S_{n+\ell} N^{n+\ell}}$) by one. In the rest of the configurations, whose probability we may estimate by unity, we get the estimate of

$$(e^{-S_{n+\ell} N^{n+\ell}} \mid \Lambda_{n+\ell} \text{ is } f\text{-normal}) \leqslant [(e^{\varpi} 2N^{2(m+\ell)})^{(1-f)}]^{N^n/N^m} e^{-\sigma(1-f) N^{-m} N^{n+\ell}} \tag{2.15}$$

Hence, we may choose larger and larger values of ℓ to dispense with the bothersome prefactors. The result, for $R \to \infty$, is that $\alpha \geqslant [1 - \eta(\ell)]\, \sigma(1-f)/N^m$ with $\eta(\ell) \to 0$ as $\ell \to \infty$. Hence we have shown

$$\alpha \geqslant \sigma((\beta) \frac{1-f}{N^m} > 0 \tag{2.16}$$

for $1 - e^{-\beta}$ above the $2d$ bond percolation threshold. ∎

Thus far, we have only used a single scale of fissures; the next step is to bring in cracks and fissures on all scales. It turns out that it is necessary to do this on two tiers, one for the surface tension itself and the other for the "percentage of fissures" i.e., the quantity $1 - f$. The following objects will be used in the forthcoming:

Definition. Let (m_k), (r_k) and (ℓ_k) denote increasing sequences of integers with $m_k > r_k$. We consider the events $F_{m_k, r_k}(w_k, \ell_k)$ which are defined as the set of configurations $(G^{(r_k)}_{m_k}, G^{(r_k)'}_{m_k})$ in $[0, 2] \times [0, 1]$ such that for all $\ell \geqslant \ell_k$, the probability in $(G^{(r_k)}_{m_k; \ell_k}, G^{(r_k)'}_{m_k; \ell_k})$ of observing any of the dual paths described in Eq. (2.11) is smaller than $\exp(-w_k N^{r_k + \ell_k})$. If μ is any number for which $e^{\mu - 1} > 2e^{\varpi}$ (the 2 is a safety factor), then, for convenience, we will denote the probability of $F_{m_k, r_k}(w_k, \ell_k)$ by $1 - e^{-\mu f_k}$.

The quantities m, r (and f) that were used in Proposition 2.5 will be the starting point for all these sequences. We will denote these quantities by m_0, r_0 and f_0. Any starting value for ℓ_0 will suffice provided that it is large compared with σ^{-1}.

Following the analysis of Proposition 2.5, if the event $F_{m_k, r_k}(w_k, \ell_k)$ occurs with probability exceeding $1 - e^{-\mu f_k}$, one can prove that in lattices $\Lambda_{n+\ell}$ (with ℓ large and $n \to \infty$) the surface tension is (essentially) at least as large as $(1 - f_k)\, w_k$ with probability tending to one exponentially fast in N^n. However, more than this can be achieved: It should be noted that in the course of the proof of Proposition 2.5, no explicit use was made of the

fact that the crossing path was between the midpoints of the sides of a square. In particular, the argument holds for any path that is forced to cross any fixed polygonal shape of scale L. [Of course a slightly different meaning will have to be assigned to the words $\mathcal{N}(K)$, but the asymptotics of the appropriate "$\mathcal{N}(K)$" will always be the same.] Hence, we can run the same argument looking for one of three types of crossings on a 2×1 rectangle. The result is:

Corollary (to Proposition 2.5). Let $F_{m_k, r_k}(w_k, \ell_k)$ be as described above. Then for any number $\tilde{f}_k > 0$, it is possible to find an $\tilde{m}_k > 0$ and an $\tilde{\ell}_k$ such that with probability larger than $1 - \exp(-\mu/\tilde{f}_k)$, the event

$$F_{\tilde{m}_k, r_k}((1-f_k)[1-\eta(\tilde{\ell}_k)]\, w_k \tilde{\ell}_k)$$

occurs, with $\eta(\tilde{\ell}_k) \to 0$ as $\tilde{\ell}_k \to \infty$.

Thus we sacrifice surface tension for a probabilistic estimate. We could, of course, use more than just the "small" $F_{m_k, r_k}(w_k, \ell_k)$ events when we go to the larger scale: Indeed, these small fissures may be combined with a large-scale crack [i.e., $F_{\tilde{m}_0, r_0}(w_0, \ell_0)$] which would give us an improved surface tension. However, we then reduce the probability factor back to the level of f_0 and, as it turns out, this cycle will never push the bound on the surface tension past the order of σ/N^{k_0}. The latter may or may not be regarded as a substantial improvement over the results of Proposition 2.5, but in any case, the conclusion is far from optimal. To break the cycle, we must use a combination of events with and without the large-scale cracks when we perform the "probabilistic" stage of the rescaling. This will allow us to keep the surface tension very near its improved value as we drive the probability up to unity. Keeping these ideas in mind, we are ready for the following result:

Theorem 2.6. Consider the percolation/aerogel system with bond parameter $1 - e^{-\beta}$ in excess of the percolation threshold and let $\sigma(\beta)$ denote the surface tension of the percolation system. Then, for any $Q < Q_c$, $\alpha(p, Q, \beta) = \sigma(\beta)$ for all $p \in [0, 1]$.

Proof. Let (f_k) denote a decreasing sequence with $f_k \geqslant 0$ and $\lim_{k \to \infty} f_k = 0$. We will take $f_0 = f$ from Proposition 2.5. For integers m_k, r_k, ℓ_k (with $m_k \geqslant r_k$) and numbers $w_k \leqslant \sigma$, let $F_{m_k, r_k}(w_k, \ell_k)$ denote the above-described events. Then we claim that there are sequences (m_k), (r_k), (ℓ_k) and (w_k) such that with probability exceeding $1 - e^{-\mu/f_k}$ the events $F_{m_k, r_k}(w_k, \ell_k)$ occur and that for these (w_k), $\lim_{k \to \infty} w_k = \sigma$.

We are in good shape if $k = 0$. Suppose then that an $F_{m_k, r_k}(w_k, \ell_k)$ has been produced with a probability that exceeds $1 - e^{-\mu/f_k}$. Let f_{k+1} be

specified and let d_k be some small number which, to be definite, we will assume satisfies $d_k \leqslant f_k$. Finally, let $n_k \gg m_k$ and $\ell_{k+1} > \ell_k$ denote integers, the precise values of which will be specified later. First, we will consider the 2×1 rectangular lattices of scale $N^{n_k + \ell_{k+1}}$; however, we will only look at the configurations $(G^{(r_k)}_{n_k;\ell_{k+1}}, G^{(r_k)'}_{n_k;\ell_{k+1}})$. Using the 2×1 rectangles of side N^{m_k}, most of which contain various versions of the event $F_{m_k, r_k}(w_k, \ell_k)$, we get, according to the Corollary to Proposition 2.5, the event $F_{n_k, r_k}((1-\eta)(1-f_k)w_k, \ell_{k+1})$ with probability tending to one exponentially fast in the lattice size, where $\eta = \eta(\ell_{k+1}) \to 0$ as $\ell_{k+1} \to \infty$. Let n_k be chosen large enough so that this probability exceeds $1 - e^{-\mu/d_k}$. So far, nothing beyond Proposition 2.5 has been achieved; however, along the way, we proceeded by establishing f_k-normalcy for the configurations $(G^{(r_k)}_{n_k}, G^{(r_k)'}_{n_k})$. [Recall the definitions from Proposition 2.5: a configuration $(G^{(r_k)}_{n_k}, G^{(r_k)'}_{n_k})$ is f_k-normal if all the spanning clusters consisting of 2×1 rectangles of scale N^{m_k} have a fraction at least as large as $1 - f_k$ of their constituents in which the analog of the event $F_{m_k, r_k}(w_k, \ell_k)$ occurs. In this context, a "spanning cluster" of the $2N^{m_k} \times N^{m_k}$ lattice means from the top to the bottom or from the left side to the midline or from the right side to the midline.] Now let us consider the consequences in the $2N^{n_k} \times N^{n_k}$ lattice if the configuration is both f_k-normal and (the original) event $F_{m,r}$ [which, of course, implies the event $F_{m_0, r_0}(w_0, \ell_0)$] occurs. Observe first and foremost that because $n_k \gg m_k > r_k$, the two events are disjoint. Indeed, on the unit scale, $F_{m,r}$ uses blocks of vacancy of scale larger than N^{-m} while f_k-normalcy is measured on $(G^{(r_k)}_{n_k}, G^{(r_k)'}_{n_k})$ and therefore only uses blocks of scale smaller than $N^{-(n_k - r_k)}$. Let us condition on the existence of some particular obstructing crack contributing to the event $F_{m,r}$. (Of course, the cracks must be ordered in some fashion so as to obtain a disjoint partition of the event $F_{m,r}$. As we will seen the particulars of the crack do not enter into the argument.) Now let us suppose that on the lattice $\Lambda_{n_k + \ell_{k+1}} \cup \Lambda'_{n_k + \ell_{k+1}}$, there is dual crossing of the type described in Eq. (2.11). As usual, we track the path into its connected cluster of 2×1 rectangles of side $N^{m_k + \ell_{k+1}}$ lattice spacings. Any spanning cluster must have a certain minimum number of constituents in the crack and it is seen that each of these will contribute a factor on the order of $\exp(-\sigma N^{m_k + \ell_{k+1}})$ to the overall estimate on the probability of observing the particular cluster event. To be precise, if $\sigma' < \sigma$, then for ℓ_{k+1} sufficiently large, at least $[(1/N^m \times N^{n_k - m_k}] - 2$ of the rectangles in any spanning cluster may be estimated by the factor $\exp(-\sigma' N^{m_k + \ell_{k+1}})$. (The -2 is because the first rectangle entering and the last rectangle leaving the crack may be half in and half out of the crack.) For a particular spanning cluster $\mathscr{C}$, with $|\mathscr{C}| = K$, let us say that a fraction $\mathfrak{s}$ has "fallen into the crack." Assuming the worst case scenario, of the remaining $(1-\mathfrak{s})K$ rectangles, in at least $(1 - \mathfrak{s} - f_k)K$ of the cases, the

analog of the event $F_{m_k, r_k}(w_k, \ell_k)$ occurs. Hence these will contribute a net decay factor of $[\exp(-w_k N^{m_k+\ell_{k+1}})]^{[1-s-f_k]K}$. Overall, we get

$$\begin{aligned}
&\text{Prob}(\mathscr{C} \mid \Lambda_{n_k+\ell_{k+1}} \cup \Lambda'_{n_k+\ell_{k+1}} \text{ is } f_k\text{-normal} \cap F_{m,k}) \\
&\quad \leqslant \exp\{-KN^{m_k+\ell_{k+1}}[\sigma' s + w_k(1-s-f_k)]\} \\
&\quad \leqslant \exp(+\sigma' N^{m_k+\ell_{k+1}}) \exp\left(-\frac{\sigma'+w_k}{N^m} N^{n_k+\ell_{k+1}}\right) \\
&\quad \times \exp[-(w_k N^{m_k+\ell_{k+1}}) K(1-f_k)] \qquad (2.17)
\end{aligned}$$

Summing over all K, it is seen that under the condition in Eq. (2.17), we have achieved the event $F_{n_k, r}(w_k^*, \ell_{k+1})$ where

$$w_k^* \geqslant \left[\frac{\sigma'-w_k}{N^m} + w_k(1-f_k)\right](1-\eta) \qquad (2.18)$$

with $\eta \to 0$ and $\sigma' \to \sigma$ as $\ell_{k+1} \to \infty$.

We remind the reader that the above mentioned $F_{n_k, r}(w_k^*, \ell_{k+1})$ was produced as a subset of the intersection of two disjoint events. In particular, these are (i) the old $F_{m,r}$, which occurs with probability exceeding $1-e^{-\mu/f_0}$, and (ii) the fact that $(G_{n_k}^{(r_k)}, G_{n_k}^{(r_k)})$ is f_k-normal, which occurs with probability exceeding $1-e^{-\mu/d_k}$. Let us temporarily denote this latter event [which, we recall, implies $F_{n_k, r_k}((1-\eta)(1-f_k)(w_k), \ell_{k+1})$] by $\tilde{F}_{n_k, r_k}$. Notice that in order to achieve the estimates in Eqs. (2.17) and (2.18) we have used an enormous range of scales. Indeed, setting the unit scale at $N^{\ell_{k+1}}$ (so as not to count the ℓ_{k+1} "artificial" subdivisions), we are working with a total of $n_k - r$ different sizes of vacancies. We are now ready for the final step.

Let $m_{k+1} \gg n_k$ and define $r_{k+1} = n_k - r$. On the (huge) $2N^{m_{k+1}} \times N^{m_{k+1}}$ lattice, consider the preceding sorts of subdivisions into 2×1 rectangles of scale N^{n_k} which, in turn, form spanning clusters, etc. A configuration $(G_{m_{k+1}}^{(r_{k+1})}, G_{m_{k+1}}^{(r_{k+1})'})$ will be called $f_0 \circ d_k$-normal if in each spanning cluster $\mathscr{C}$, a fraction at least as large as $1-f_0$ of the constituent rectangles enjoy the analog of the event $F_{m,r}$ and a faction at least as large as $1-d_k$ of them enjoy the analog of the event $\tilde{F}_{n_k, r_k}$. According to our routine estimates, the probability of $f_0 \circ d_k$-normalcy tends to unity exponentially fast in the scale of the huge lattice, $N^{m_{k+1}}$. Let us choose m_{k+1} large enough so that this probability exceeds $1-e^{-\mu/f_{k+1}}$. We will now estimate the surface tension under the condition that $G_{m_{k+1}}^{(r_{k+1})}, G_{m_{k+1}}^{(r_{k+1})'}$ is $f_0 \circ d_k$-normal. In the worst case scenario, each cluster $\mathscr{C}$ with $|\mathscr{C}| = K$, has at least $(1-f_0-d_k)K$ of the constituent rectangles that experience both events, $f_0 K$ getting the event

$\tilde{F}_{n_k, r_k}$ alone, and $d_k K$ that get nothing at all. As discussed earlier, the former event implies the analog of the event $F_{n_k, r}(w_k^*, \ell_{k+1})$ in these rectangles, and hence a total contribution to the decay of $\exp[-(w_k^* N^{n_k+\ell_{k+1}})(1-f_0-d_k)]$. The latter will give us an additional factor of $\exp\{-(1-\eta)[w_k(1-f_k)]\, N^{n_k+\ell_{k+1}} f_0 K\}$. Summing over all K (and following through with the final few steps of Proposition 2.5) gives us our $(k+1)$th estimate on the surface tension, w_{k+1}:

$$\begin{aligned} w_{k+1} &\geqslant (1-\eta)[(1-f_0-d_k)\, w_k^* + f_0(1-f_k)(1-\eta)\, w_k] \\ &= [1-\eta(\ell_k)][1-\eta(\ell_{k+1})] \\ &\quad \times \left[(1-f_k)(1-d_k)\, w_k + (1-f_0-d_k)\left\{\frac{\sigma' - w_k}{N^m}\right\}\right] \end{aligned} \tag{2.19}$$

where $\eta(\ell_k) \to 0$ and $\sigma' \to \sigma$ if we let $\ell_k \to \infty$. Under these circumstances, it is not difficult to see that the sequence, (w_k), converges to $\sigma(\beta)$ and, at last, the proof is complete. ∎

Corollary (to Theorem 2.6). Under the conditions of Theorem 2.6,

$$\lim_{k \to \infty} \alpha^*_{[k]} = \sigma(\beta)$$

Proof. Following the discussion in the Corollary to Proposition 2.5, it is seen that the quantities $w_k(1-f_k)[1-\eta(\ell_{k+1})]$ actually provide a lower bound on the surface tensions $\alpha^*_{[r_k+\ell_{k+1}]}$. The desired result therefore follows From the above analysis. ∎

The statement of Theorem 2.7 is almost identical to that of Theorem 2.6 except with regard to the identification of the transition temperature in the uniform system. At the beginning of Appendix B is a finite-size scaling criterion for low-temperature behavior and a definition of a transition temperature β_t^{-1}. Although at present there is no proof that this definition coincides with the usual definition of the transition temperature—except for percolation—it is difficult to believe that this is not the case. (Most likely, a proof could be provided for $q=2$ and integer $q \gg 1$ with currently available techniques.) In any case, it is clear that some transition occurs at β_t and that true low-temperature behavior only occurs if $\beta > \beta_t$. Hence, modulo a precise definition of this quantity, we may state the following result.

Theorem 2.7. Consider the q-state random cluster aerogel systems with $q \geqslant 1$, $Q < Q_c$, and bond density parameter $1-e^{-\beta}$. If β exceeds β_t [defined in Eq. (B.1)] then for all $p \in [0, 1]$, we have $\alpha = \alpha^* = \sigma(\beta)$ where $\sigma(\beta)$ is the surface tension for the uniform q-state random cluster system.

Concluding Remarks

The fact of Theorem 2.7, especially in the region $p > p_G(Q_c)$, leads to an interesting and disturbing set of observations. We have at our disposal all the quantities α_R and $\alpha^*_{[k]}$. Now different behavior for the different $\alpha^*_{[k]}$'s is of course to be expected since these quantities actually pertain to different models. Indeed, as we approach the critical curve From the high-temperature side, the beginning portion of the sequence will vanish. However at least in the region specified in Theorem 2.6, for large k, $\alpha^*_{[k]} \approx \sigma(\beta)$. The anticipated behavior of the α_R is another story altogether. According to the usual scaling pictures, all of the α_R's—as well as the limiting α—should exhibit qualitatively similar behavior as a phase boundary is approached. We will demonstrate that the above fails dramatically as we approach the vertical portion of the phase boundary

In this regard, let us recapitulate that by holding p and β fixed (at sufficiently large values) and allowing Q to vary, we get discontinuous behavior at Q_c for $\alpha(Q)$. We will now show that for any finite R, $\alpha_R \downarrow 0$ as $Q \uparrow Q_c$. Recall the discussion of the finite-size scaling length $N^{k_0(Q)}$: This is defined by the smallest k_0 such that with some definite probability, of order unity, one can observe a pore crossing, the hard way, across a 2×1 rectangle that houses two independent copies of A_{k_0}. For convenience, let us modify this definition to read "the easy way across"; we may denote the corresponding k by $\tilde{k}_0$.[6] A central result in ref. 6 was that as the sol–gel phase boundary is approached from the sol side, the quantity k_0—and hence $\tilde{k}_0$—is divergent. Thus at the scale $N^{\tilde{k}_0}$, practically by definition, there is a long-way crossing of the 2×1 rectangles with probability larger than, say, e^{-a}. Let us turn our attention to Λ_n. If we just focus on the $\tilde{k}_0$ smallest scales, we will observe a left–right gel crossing inside the strip of width $N^{\tilde{k}_0}$ surrounding the midline with probability larger than $\exp\{-2aN^n/N^{\tilde{k}_0}\}$. This connection is hooked up to the midpoints with probability that is uniform in n. Finally, it is noted that the band of width $\tilde{k}_0$ centered around the midline does not lie inside any pores of scale larger than $N^{\tilde{k}_0}$ with probability

$$(Q^{N^n/N^{\tilde{k}_0}-1})(Q^{N^n/N^{\tilde{k}_0}-2})(\cdots)(Q^N).$$

[6] It is not hard to show that the ratio $k_0/\tilde{k}_0$ is uniformly bounded for (p, Q) in any closed subset of $(0, 1)^2$. For multiscale percolation problems involving pore events, these sorts of things are easily proved, cf. ref. 4, Lemma 6.2. In single-scale problems, the results do not come so easily.

Putting these facts together, it has just been demonstrated that the surface tension is zero with probability larger that $\exp\{-bN^n/N^{\tilde{k}_0}\}$ for some number b that is uniformly bounded. Evidently, for any R,

$$\mathbf{E}\left(\exp\left(-\frac{1}{R}S_nN^n\right)\right)\geqslant\exp\left(-b\frac{N^n}{N^{\tilde{k}_0}}\right) \tag{2.20}$$

so that $\alpha_R\leqslant bR/N^{\tilde{k}_0}$. Hence, all the α_R's tend to zero, which indicates (or proves) that this is a genuine critical phenomenon. (In particular, α_R, for *finite* R appears to have a direct interpertation as a correlation length.) It may well be the case that in other portions of the phase diagram, the limiting surface tension will also exhibit continuous behavior.

APPENDIX A. STABILITY OF THE FREE ENERGY

Recall from our preliminary discussion that the model is confined to the unit square and that at the nth stage, this represents a lattice Λ_n, of spacing N^{-n} consisting of N^{2n} sites. Of course, with large probability, the vast majority of these sites come equipped with spin variables and the vast majority of these reside in nearly uniform environments. In this light, it is difficult to imagine that any limiting statistical mechanics (i.e., the distributional behavior of local observables) could end up differently from that which is possible for the corresponding uniform system. We will not delve into a formal proof (or worse yet, a formal definition) of the above assertion. Instead, we will be content with a proof that the preceding holds for all thermodynamic quantities. In particular, for finite n, there is always a free energy per site (which is a random variable). Not surprisingly, a limiting free energy exists, and is equal to that of the corresponding uniform system. This will be the subject of Theorem A.1, which follows some preliminary definitions and notation.

Definitions. Let s denote a generic spin variable which, for simplicity, we assume takes on values in a compact space. Let $d\mu(s)$ denote the single-spin (or *a priori*) distribution and let us assume that the total size of the single-spin space as measured by μ is finite. Let $J(\cdot,\cdot)$ denote the interaction energy function for a pair of spins and let $h(s)$ denote the magnetic energy for a single spin. It is assumed that these functions are bounded. On the square lattice, the Hamiltonian is defined by the usual formal expression:

$$H=\sum_{\langle i,j\rangle}J(s_i,s_j)+\sum_i h(s_i) \tag{A.1}$$

where $\langle i,j\rangle$ denotes a nearest neighbor pair. For $\Lambda \subset \mathbb{Z}^2$, let s_Λ denote a spin configuration on Λ. Let $\partial\Lambda$ denote the sites in Λ^c that have a neighbor in Λ. Then, for any fixed configuration of boundary spins $s_{\partial\Lambda}$, the partition function on Λ, at inverse temperature β is given by

$$\mathscr{Z}_{\Lambda;\beta}(s_{\partial\Lambda}) = \int \prod_{j\in\Lambda} d\mu(s_j) \exp\{-\beta H(s_\Lambda, s_{\partial\Lambda})\} \tag{A.2}$$

where $H(s_\Lambda, s_{\partial\Lambda})$ is notation for the finite sum

$$H(s_\Lambda, s_{\partial\Lambda}) = \sum_{\langle i,j\rangle \text{ in } \Lambda} J(s_i, s_j) + \sum_{i\in\Lambda} h(s_i) + \sum_{\substack{\langle i,j\rangle \\ i\in\Lambda,\, j\in\partial\Lambda}} J(s_i, s_j) \tag{A.3}$$

If $K_n \subset \Lambda_n$ represents a configuration of pores (not necessarily generated by the aerogel process), we will denote by $\mathscr{Z}_{K_n;\beta}(s_{\partial\Lambda_n})$ the partition function defined as in Eq. (A.2) and (A.3) but with all terms involving any $j \in C_n$ ($\equiv \Lambda_n \backslash K_n$) omitted. Finally, we will denote by $\mathscr{Z}^{[+]}_{\Lambda_n;\beta}$ the maximum value for the partition function that can be achieved by adjusting the the spin configuration on the boundary and by $\mathscr{Z}^{[+][+]}_{\Lambda_n;\beta}$ the maximum partition function that can be obtained on any subset of Λ_n:

$$\mathscr{Z}^{[+][+]}_{\Lambda_n;\beta} = \max_{K_n} \max_{s_{\partial\Lambda_n}} \mathscr{Z}_{K_n;\beta}(s_{\partial\Lambda_n}).$$

We will also use $\mathscr{Z}^{[-]}_{\Lambda_n;\beta}$ and $\mathscr{Z}^{[-][-]}_{\Lambda_n;\beta}$ as notation for similarly defined quantities with the word maximum replaced by minimum.

We are ready to prove Theorem A.1.

Theorem A.1. Let $f(\beta)$ denote the free energy for the $2d$ spin system as defined by the Hamiltonian in Eq. (A.1) and let (C_n) denote a random sequence of aerogels generated as described in Section 1. Then for any sequence of boundary conditions $(s_{\partial\Lambda_n})$ und for all β, p and Q,

$$\lim_{n\to\infty} -\frac{\log \mathscr{Z}_{K_n;\beta}(s_{\partial\Lambda_n})}{N^{2n}} = f(\beta)$$

with probability one.

Proof. By the standard "existence of thermodynamics" arguments, it is clear that

$$\lim_{n\to\infty} \frac{\log \mathscr{Z}^{[+]}_{\Lambda_n;\beta}}{N^{2n}} = \lim_{n\to\infty} \frac{\log \mathscr{Z}^{[-]}_{\Lambda_n;\beta}}{N^{2n}} = f(\beta) \tag{A.4}$$

Furthermore, by the boundedness properties of the Hamiltonian mentioned just prior to Eq. (A.1), it is easily seen that

$$e^{FN^n} \geqslant \mathscr{Z}^{[+][+]}_{\Lambda_n;\beta} \geqslant \mathscr{Z}^{[-][-]}_{\Lambda_n;\beta} \geqslant e^{-FN^n} \tag{A.5}$$

holds uniformly in n for some finite constant F. For any sequence of aerogels (A_{n-1}, C_n), let $|A_{n-1}|$ denote the number of squares that had been considered alive just before the nth step of the process. For any boundary condition and any n and m with $m > n$ we may write

$$\mathscr{Z}_{K_m;\beta}(s_{\partial\Lambda_n}) \leqslant [\mathscr{Z}^{[+][+]}_{\Lambda_{m-n};\beta}]^{|A_n|} [\mathscr{Z}^{[+]}_{\Lambda_{m-n};\beta}]^{N^{2n}-|A_n|} \tag{A.6}$$

with a corresponding lower bound in which the plusses are replaced by minuses. But then

$$\frac{\log \mathscr{Z}_{K_m;\beta}}{N^{2m}} \leqslant \frac{F|A_n|}{N^{2n}} + \left(1 - \frac{|A_n|}{N^{2n}}\right) \frac{\log \mathscr{Z}^{[+]}_{\Lambda_{n-m};\beta}}{N^{2(m-n)}} \tag{A.7}$$

along with a similar looking lower bound. Hence, as $m \to \infty$, the right hand side of Eq. (A.7) converges to

$$F\left[\frac{|A_n|}{N^{2n}}\right] - \left(1 - \left[\frac{|A_n|}{N^{2n}}\right]\right) f(\beta)$$

Next, observe that $|A_n|$ is distributed like a branching process with a mean of QN^2 out of a maximum possible N^2 progeny. Therefore we have that for any $Q < 1$, $|A_n|/N^{2n} \to 0$ with probability one and thus

$$\lim_{m\to\infty} \frac{\log \mathscr{Z}_{K_m;\beta}}{N^{2m}} \leqslant -f(\beta) \tag{A.8a}$$

with probability one. Similar considerations show that

$$\lim_{m\to\infty} \frac{\log \mathscr{Z}_{K_m;\beta}}{N^{2m}} \geqslant -f(\beta) \tag{A.8b}$$

with probability one. ∎

Remark. We observe that the above proof extends, with almost no modifications, to more general systems, including the d-dimensional versions of the aerogel process.

APPENDIX B. SOME RESULTS ON THE LOW-TEMPERATURE PHASE OF THE POTTS MODELS AND A PROOF OF THEOREM 2.7

In this appendix, we provide a proof of Theorem 2.7, namely if $Q < Q_c$ and the temperature is below the transition temperature of the uniform q-state Potts system, then the aerogel system for this β, Q (and q) has, for all $p \in [0, 1]$, the same surface tension as the counterpart uniform system. In fact, all of the above stated will be established for the q-valued random cluster models with $q \geqslant 1$.

Our starting point will be a precise definition of what we mean by the low-temperature phase for the uniform system. Since most of this appendix will focus on exponential decay of various connectivity (or correlation) functions, it is more convenient to work with the dual variables. Thus, the low-temperature phase will really be the high-temperature phase for the dual model. The reader should be cautioned that the words wired and free will therefore be exchanged. To avoid confusion, we will desist from the usage of the words "surface tension" until we are ready for the proof of Theorem 2.7. However, we will still use σ to indicate the corresponding rate of decay and we will use λ, as defined just prior to Eq. (1.5*), to denote the bond density parameter.

Definition. Consider the $2L \times L$ lattice $\{(x_1, x_2) \in \mathbb{Z}^2 | 0 \leqslant x_1 \leqslant 2L, 0 \leqslant x_2 \leqslant L\}$ on which is defined a q-valued random cluster model with $q \geqslant 1$ and bond density parameter λ. Let ε_L denote the random cluster probability, in the ensemble with wired boundary conditions, of observing a top–bottom crossing of the lattice by open bonds or a crossing from the left side ($x_1 = 0$) to the middle ($x_1 = L$) by open bonds or a crossing from the middle ($x_1 = L + 1$) to the right side ($x_1 = 2L$) by open bonds. [See Fig. 5. Notice that this is the exact analog of the event described in (2.11).] We define

$$\lambda_t = \sup\{\lambda \mid \lim_{L \to \infty} \varepsilon_L = 0\} \tag{B.1}$$

The inverse temperature corresponding to this value of λ is denoted by β_t.

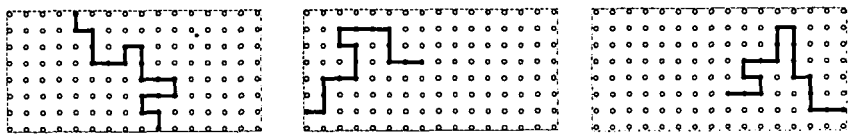

Fig. 5. Dual paths contributing to ε_L;

Remark. There is no obvious subadditive-type inequality relating the quantities ε_L. This is not due to the composite nature of the event; the same remark holds for the ordinary crossing probabilities with these boundary conditions. In particular, unlike the quantity σ—which the reader will recall is here defined with free boundary conditions—our wired boundary conditions completely disrupt all the standard procedures. Nevertheless, the following can be established:

Lemma B.1. The limit

$$e^{-\tilde{\sigma}} = \lim_{L \to \infty} (\varepsilon_L)^{1/L}$$

exists. In, particular, $\lambda < \lambda_t$ if and only if $\tilde{\sigma}$ is positive.

Proof. Most of the above sentence is a direct consequence of the forthcoming finite-size scaling argument: We claim that there is an $\varepsilon > 0$ such that if for any L_0, $\varepsilon_{L_0} < \varepsilon$, then $\lim_{L \to \infty} \varepsilon_L = 0$ and, in particular, $\limsup_{L \to \infty} (\varepsilon_L)^{1/L} < 1$. (And, in fact, is of the order of e^{-1/L_0}.)

The claim is established using an argument, that is little more than a recapitulation of Proposition 2.5; however, the derivation is considerably simpler because we are dealing with a uniform system. Suppose that $\varepsilon_{L_0} < \varepsilon$ with ε a number to be specified later. Let $L \gg L_0$ and consider the event of one of the crossings described in the definition of ε_L. For simplicity, let us assume that L is a multiple of L_0 (otherwise, we will just cut down on the size of the lattice). As in the proof of Proposition 2.5, we may track a path according to which $2L_0 \times L_0$ rectangles on a grid of scale L_0 are threaded by the path. Further, we partition the event in question according to which (unordered) cluster is the one threaded. As a result, we get

$$\varepsilon_L \leqslant \sum_{K > L/L_0} \mathcal{N}(K)(\varepsilon_{L_0})^K \tag{B.2}$$

where, as in Proposition 2.5 and Theorem 2.6, we are a little loose with the definition of $\mathcal{N}(K)$; we agree that it stands for the number of appropriate spanning clusters of size K and that the $K \to \infty$ limit of $[N(K)]^{1/K}$, e^{ϖ}, exists and is finite. It is evident that if ε is any number that is smaller than this limit, then the limit of ε_L is zero. Under these circumstances, if $L_2 \gg L_1$ with L_1 sufficiently large we can also write

$$(\varepsilon_{L_2})^{1/L_2} \leqslant \text{const} \cdot e^{\varpi'})^{1/L_2} (\varepsilon_{L_1})^{1/L_1} \tag{B.3}$$

where ϖ' is any number larger than ϖ. Equation (B.2) allows us to establish the existence of $e^{-\tilde{\sigma}} \equiv \lim_{L \to \infty} (\varepsilon_L)^{1/L}$ with $\tilde{\sigma} \geqslant 0$.

Thus, if $\lambda < \lambda_t$ (so that $\varepsilon_L \to 0$) we have demonstrated that $\tilde{\sigma}$ exists and is positive. Conversely, if $\tilde{\sigma} > 0$ exists, we have *a priori* that $\lambda \leqslant \lambda_t$. But then, for some large L, ε_L is still smaller than E even at a slightly larger value of the bond density parameter. This shows that the λ in question must have been strictly smaller than λ_t. Evidently, at λ_t, $\varepsilon_L > \varepsilon$ for all L and here we may state that $\tilde{\sigma}$ is zero. ∎

Corollary. If $\lambda < \lambda_t$, the phase is unique

Proof. Let V_L denote the boxes of side L centered at the origin, let $L' \gg L$, and consider the free and wired measures of $V_{L'}$. Since the restriction to V_L of any limiting measure lies between these two (in the sense of FKQ), it is sufficient to show that as $L' \to \infty$, the restriction of these two measures merges.

Now if $\lambda < \lambda_t$, it is not difficult to show that, even with wired conditions on $V_{L'}$, the probability of a connection between ∂V_L and $V_{L'}$ tends to zero at (approximately) the rate of $e^{-\tilde{\sigma}(L'-L)}$. In the absence of such a connection, the wired measure on $V_{L'}$ is FKG subdominant to the free one. (This is because the lack of connection can be represented by the presence of a circuit of "dual" bonds separating $\partial V_{L'}$ from V_L which, in turn, is equivalent to free boundary conditions closer to V_L than $\partial V_{L'}$.) From this, uniqueness can be easily established. Furthermore, for integer $q \geqslant 2$, this easily translates into uniqueness for the corresponding (high-temperature) spin system. ∎

Remark. The proof of Theorem 2.7 would be considerably simpler if it were possible to establish that $\tilde{\sigma} = \sigma$. In this case, we could essentially plug into the proof of Theorem 2.6. Unfortunately, we believe that in general this is not the case. (Although it certainly holds for percolation and possibly the Ising magnet.) Indeed, it is conceivable that in a half-space q-state system with wired boundary conditions there is a surface layer where the appearance of the system is quite different from that of the bulk phase. This, in the language of the FK representation, could result in a "pinning" of a path with a corresponding lowering of the associated rate of decay. Under such circumstances, wired boxes may be crossed by paths that "creep along the boundary" and the probability of these crossings will then go to zero at a rate constant which is slower than that provided by the bulk. Thus, even for the analysis of the uniform system, we will have to develop a box-crossing technology where the paths are forced to stay away from the boundaries

On a happier note, at least the bulk correlation length can be identified with the quantity σ^{-1}. Although this result is fairly well known, we will provide a proof for the sake of completeness.

We begin by defining $G_{x,y}$ to be the probability in the (unique) infinite volume state that the points x and y belong to the same connected cluster. We will use the notation m for the limit

$$m = \lim_{L \to \infty} -\frac{1}{L} \log G_{\mathbf{0},(0,L)} \tag{B.4}$$

It is clear that $\sigma \geqslant m$ ($\geqslant \tilde{\sigma}$); we now prove the following result.

Lemma B.2. For $\lambda < \lambda_t$, $m = \sigma$.

Proof. Let $T_{x,y}$ denote the event that x is in the connected component of y and recall the lattices V_L defined by $\{x | |x_1|, |x_2| \leqslant L/2\}$. It is assumed, for simplicity, that L is even. If x and y are in V_L we will denote the probability of $T_{x,y}$ with free boundary conditions on V_L by $G^{(f)}_{x,y}$; the L-dependence will be clear from context.

In this language, the quantity σ is given by

$$\sigma = \lim_{L \to \infty} -\frac{1}{L} \log G^{(f)}_{(-L/2,0),(+L/2,0)} \tag{B.5}$$

Let us further assume that L is of the form nK with K large and, for the moment, fixed. We can write

$$G^{(f)}_{(-L/2,0),(+L/2,0)} \geqslant G^{(f)}_{(-L/2,0),(-L/2+K,0)} \times G^{(f)}_{(-L/2+K,0),(-L/2+(K+1),0)} \cdots G^{(f)}_{(+L/2-K,0),(+L/2,0)} \tag{B.6}$$

Let us focus attention on the rth [or $(n-r)$th] term in the above product: $G^{(f)}_{L/2-rK,0),(L/2-(r-1)K,0)}$. We claim that except for the first and last few terms, these probabilities differ only slightly from the corresponding probabilities in the infinite-volume state:

$$G_{L/2-rK,0),(L/2-(r-1)K,0} \equiv G_{\mathbf{0},(K,0)}$$

To see this, let $\tilde{R}_r$ denote the event that there is a connection between one of the points $(L/2-(r-1)K, 0)$ or $(L/2-rK, 0)$ and the boundary ∂V_L. It is not hard to show that, in the infinite- volume state,

$$\text{Prob}(\tilde{R}_r) \leqslant \text{const} \cdot L^2 e^{-m[L/2-(r-1)K]}$$

Now observe that the infinite state conditional probability

$$\text{Prob}(T_{(L/2-rK,0),(L/2-(r-1)K,0)} \mid \tilde{R}_r^c)$$

is actually smaller than $G^{(f)}_{(L/2-rK,\,0),\,(L/2-(r-1)K,\,0)}$. Indeed this conditioning necessarily implies the presence of boundary conditions less favorable to the event in question. Thus we may write

$$G^{(f)}_{(-L/2,\,0),\,(+L/2,\,0)} \geqslant \prod_r D_r \tag{B.7}$$

where $D_r = G_{\mathbf{0},\,(K,\,0)} - \mathrm{Prob}(\tilde{R}_r)$ if the latter is positive and is just the rth term on the right-hand side of Eq. (B.6) for the few terms where this happens to be negative. The result as $L \to \infty$, is

$$\sigma \leqslant \frac{-\log G_{\mathbf{0},\,(K,\,0)}}{K} \tag{B.8}$$

which implies $m \geqslant \sigma$ and hence that $m = \sigma$. ∎

We now launch into our program designed to keep the paths away from the boundary.

Definition. Let T be some integer larger than two. We define a string cluster of length T on $\mathbb{Z}^2$ to be a connected cluster of size T such that each site in the string has at most two neighbors in the cluster, and two distinguished sites—the endpoints—have but one neighbor in the cluster. Observe that this is slightly more restrictive than the sites of a SAW since here no doubling back is permitted. Let $\mathfrak{C}_1, \ldots, \mathfrak{C}_{v_T}$ denote all the string clusters of length T that contain the origin, and let $\mathfrak{C}_{j,L}$ denote the lattice where each site of $\mathfrak{C}_j$ is replaced by an $L \times L$ square. We will consider random cluster problems on the $\mathfrak{C}_{j,L}$ with wired boundary conditions. For any j, consider the lattice $\delta\mathfrak{C}_{j,L}$, which is $\mathfrak{C}_{j,L}$ with its endboxes lopped off. If, in $\delta\mathfrak{C}_{j,L}$, we neglect that portion of the boundary that used to be in the interior, the remainder of the boundary is divided into two disjoint components. We will denote these by $\partial_A \delta\mathfrak{C}_{j,L}$ and $\partial_B \delta\mathfrak{C}_{j,L}$ respectively. The reader may wish to consult Fig. 6 for an illustration of the above-mentioned objects.

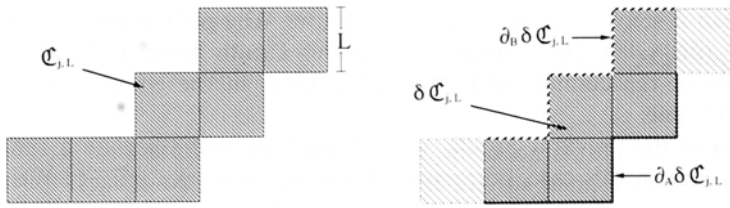

Fig. 6. $\mathfrak{C}_{j,L}$ and related constructs.

For the FK problem on the lattice $\mathbb{C}_{j,L}$, we define $\mathscr{B}_{j,L}$ to be the event that either:

(i) There is a path of occupied bonds connecting the two endboxes.

Or:

(ii) There is a path of occupied bonds in $\delta\mathbb{C}_{j,L}$ connecting $\partial_A\delta\mathbb{C}_{j,L}$ with $\partial_B\delta\mathbb{C}_{j,L}$.

We define

$$\zeta_L^T = \min_{1 \leqslant j \leqslant v_T} \text{Prob}(\mathscr{B}_{j,L}) \tag{B.9}$$

where the above probability is measured with respect to the wired states on the associated lattice $\mathbb{C}_{j,L}$. It is observed that both (i) and (ii) are determined by the configuration in $\delta\mathbb{C}_{j,L}$.

We note that for $\lambda < \lambda_r$, $\zeta_L^T \to 0$; indeed $\zeta_L \lesssim e^{-\tilde{\sigma}L}$. However, because (for big enough T) the paths are required to be long or are not allowed to creep along the boundary, we get a far more desirable result.

Proposition B.3. Consider the q-state random cluster models with $q \geqslant 1$ and $\lambda < \lambda_r$. Let T be any integer, assumed for simplicity to be even, that is large enough to ensure $(T/2-1)\tilde{\sigma} > \sigma$. Then

$$\lim_{L\to\infty} (\zeta_L^T)^{1/L} = e^{-\sigma}$$

Proof. Let T satisfy the above criteria and, in what follows, we will suppress most of the T dependence in our notation. Let $\mathbb{C}_j$ denote one of the basic shapes and let $L' \gg L \gg 1$. Note that if a crossing of type (ii) occurs on $\mathbb{C}_{j,L}$, then at least $\frac{1}{2}(T-2)$ rectangles of dimension $2L' \times L'$ must be threaded by the old definition. (Cf. Proposition 2.5 or Lemma B.1.) Thus these types of crossings will not be a major concern since they will have a probability smaller than the order of $e^{-\sigma L'}$.

In the vicinity of the two boundary regions $\partial_A(\delta\mathbb{C}_{j,L'})$ and $\partial_B(\delta\mathbb{C}_{j,L'})$, let us define two "comfort zones" which consist of all sites in $\mathbb{C}_{j,L'}$ that are less than L steps away from these boundaries. Any path between $\partial_A(\delta\mathbb{C}_{j,L'})$ and $\partial_B(\delta\mathbb{C}_{j,L'})$ in $\delta\mathbb{C}_{j,L'}$ necessarily implies a path between the two components of the comfort zone that takes place in the complement of the comfort zone.

As in the previous proofs, to each path we will associate a spanning cluster, but in this case, comprised of the shapes $\mathbb{C}_{1,L}, \ldots, \mathbb{C}_{v_T,L}$. We again conveniently assume that L' is a multiple of L, and divide $\mathbb{C}_{j,L'}$ into $L \times L$ boxes. Once the path moves away from the comfort zone associated with

$\partial_A(\delta\mathfrak{C}_{j,L'})$, one possibility is that before it breaks through the layer of $L \times L$ boxes blanketing this comfort zone, it will wander inside a string of $T-1$ boxes. In this case, we have observed a path of type (i) in a translation of some $\mathfrak{C}_{k,L}$. The other possibility is that it breaks through this layer without having achieved the above dubious accomplishment. In this case we can find a string of T boxes $\mathfrak{C}_{k,L}$ such that a path of type (ii) has occurred on this lattice. Of course in this latter case (and to a lesser extent in the former case) we may have some leeway in our choice of which boxes constitute our collection $\mathfrak{C}_{k,L}$; now and in the future, a definitive choice will be made according to some deterministic rule. It is noted that in order to perform this task, we may have picked a $\mathfrak{C}_{k,L}$ with one of its endboxes in the comfort zone or in one of the (big) endboxes $\mathfrak{C}_{j,L'} \backslash \delta\mathfrak{C}_{j,L'}$. This is fine because, by definition of the full event under consideration, the path itself must stay out of this region.

After the ith stage of the construction has been completed, we adjoin the $(i+1)$th string pretty much in accord with the procedure of Proposition 2.5: We construct the $\mathbb{Z}^2$ boundary of the existing cluster (where the $L \times L$ squares are now regarded as sites and the relevant notion of connectivity is $*$-connectedness) and intersect this object with $\mathfrak{C}_{i,L'}$. The next shape to join the cluster will be a string of length T chosen out of this string boundary.

Using the same logic that was used in Proposition 2.5, we see that the forward progress at each stage is no more than L units. Indeed the external boundary is located inside or on the smallest rectangle that can be drawn outside the cluster.

We are now ready to track the path as it continues on its journey across $\delta\mathfrak{C}_{j,L'}$. Either it wanders through $T-1$ successive boxes in the current string boundary or (as is far more likely) it crosses this boundary before having done so. When it has achieved either of these goals, we can contain that portion of the path that has done the job in a copy of one of the $\mathfrak{C}_{k,L}$'s. Of course, needless to say, we may have to use our deterministic rule to ascertain which one.

The next few steps differ in only in minor details from those of Proposition 2.5 or Lemma B.1. The result is that we may write

$$\zeta_{L'} = \sum_{K \geqslant L'/L-2} \mathfrak{M}_T(K)[\zeta_L]^K \tag{B.10}$$

where $\mathfrak{M}_T(K)$ is the number of appropriate spanning clusters that consist of K string clusters of length T. (The -2 is for the comfort zones.) Our next step is to dispense with the issue of "cluster entropy." As usual, it is standard that $\lim_{K\to\infty}[\mathfrak{M}_T(K)]^{1/K}$ exists; thus the only question is

whether this limit is finite. It is noted that an extension of the argument that was used in the case $T=2$ may be applied. However, the general case is best dispensed with by the following: Let us count the number of distinct *shapes* that go into $\mathfrak{M}_T(K)$ and denote this by $\mathfrak{m}_T(K)$. It is obvious that $[\mathfrak{m}_T(K)]^{1/K}$ is bounded. However, we still have to account for the different ways that a given shape can be broken up into K string clusters of length T. By definition, there are v_T different types of string clusters and thus Tv_T distinct clusters that contain the origin. Hence, any given site in any given shape gives rise to at most Tv_T possibilities. Allowing for all possibilities at all of the KT sites then gives us the drastic overestimate of

$$\mathfrak{M}_T(K) \leqslant \mathfrak{m}_T(K)[Tv_T]^{TK} \tag{B.11}$$

which is sufficient for our purposes.

Next, it has to be shown that the sum in Eq. (B.10) is convergent: this input we get straight from Lemma B.1 and the fact that $\lambda < \lambda_t$. These are all the ingredients needed to show that

$$e^{-\zeta} = \lim_{L \to \infty} \zeta_L^{1/L} \tag{B.12}$$

exists. Furthermore, we have that $\tilde{\sigma} \geqslant \zeta \geqslant \sigma \equiv m$. Although we should provisionally state that ζ may have T dependence, it is not hard to modify the preceding argument, using different T's on the different scales, to show that ζ is independent of T. (Assuming, of course that T satisfies the hypothesis of this proposition.) Our final job, then, is to show that $\zeta = m$.

All things considered, this final step is not particularly difficult; the reasoning follows closely that of Lemma B.2. Let $L \gg 1$ and let us consider the event $\mathscr{B}_{i,L}$ on some lattice $\mathbb{C}_{j,L}$. As discussed before, we need only consider the paths of type (ii); let us denote the event of such a path by $\mathscr{B}_{j,L}^{(ii)}$. We divide $\mathbb{C}_{j,L}$ into top, bottom and middle "thirds," as illustrated in

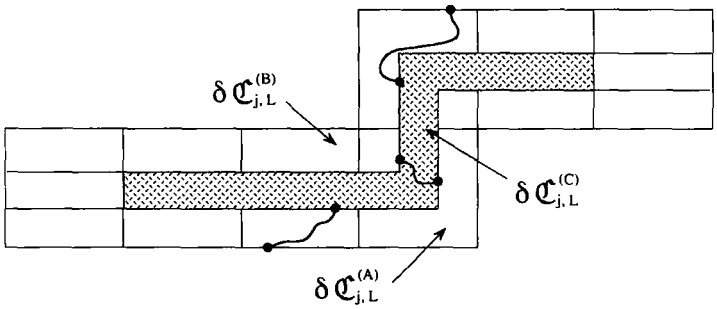

Fig. 7. Paths for the events $\mathscr{B}_{j,L}^{[A]} - \mathscr{B}_{j,L}^{[C]}$.

Fig. 7. These will be denoted by $\mathbb{C}_{j,L}^{(A)}$, $\mathbb{C}_{j,L}^{(B)}$, and $\mathbb{C}_{j,L}^{(C)}$, where, e.g., $\mathbb{C}_{j,L}^{(A)}$ has all of $\partial_A \delta\mathbb{C}_{j,L}$ in its boundary, etc. We will denote the "amputated" versions of these lattices by $\delta\mathbb{C}_{j,L}^{(A)}$, ..., $\delta\mathbb{C}_{j,L}^{(C)}$. The event $\mathscr{B}_{j,L}^{(ii)}$ is contained in the intersection of three events:

$$\mathscr{B}_{j,L}^{(ii)} \subset \mathscr{B}_{j,L}^{[A]} \cap \mathscr{B}_{j,L}^{[B]} \cap \mathscr{B}_{j,L}^{[C]} \tag{B.13}$$

where $\mathscr{B}_{j,L}^{[A]}$ is the event that there is a path in $\delta\mathbb{C}_{j,L}^{(A)}$ connecting the top and bottom portions of its boundary (the latter being the old $\partial_A \delta\mathbb{C}_{j,L}$), similarly for $\mathscr{B}_{j,L}^{[B]}$, and $\mathscr{B}_{j,L}^{[C]}$ is the event that there is a path in $\delta\mathbb{C}_{j,L}^{(C)}$, connecting the portions of its boundary that it shares with $\delta\mathbb{C}_{j,L}^{(A)}$ and $\delta\mathbb{C}_{j,L}^{(B)}$. In the final event, it is required that the path lie far away from the influence of the fixed boundary conditions on $\partial\mathbb{C}_{j,L}$. This is the key fact in the proof of the lemma.

On the basis of Eq. (B.13), we can write

$$\text{Prob}(\mathscr{B}_{j,L}^{(ii)}) \leqslant \text{Prob}(\mathscr{B}_{j,L}^{[C]})\ \text{Prob}(\mathscr{B}_{j,L}^{[B]} \cap \mathscr{B}_{j,L}^{[C]} \mid \mathscr{B}_{j,L}^{[C]}) \tag{B.14}$$

where Prob($-$) refers to the wired ensemble on $\mathbb{C}_{j,L}$.

As an upper bound, the conditioning for the event $\mathscr{B}_{j,L}^{[B]}\mathscr{B}_{j,L}^{[C]}$ can be replaced by wired boundary conditions on $\mathbb{C}_{j,L}^{(A)}$ and $\mathbb{C}_{j,L}^{(B)}$. If we do this, the probability factors and, in fact can be bounded above by a $\zeta_{L/3}^{T'}$ and a $\zeta_{L/3}^{T''}$ with T', $T'' \approx 3T$. This leaves us to deal with the term Prob($\mathscr{B}_{j,L}^{[C]}$). Consider the event $\mathscr{B}_{j,L}^{[\diamond]}$, that there is a connection between the boundaries $\partial\mathbb{C}_{j,L}$ and $\delta\mathbb{C}_{j,L}^{(C)}$. We can, of course, write

$$\begin{aligned}\text{Prob}(\mathscr{B}_{j,L}^{[C]}) = {} & \text{Prob}(\mathscr{B}_{j,L}^{[\diamond]})\ \text{Prob}(\mathscr{B}_{j,L}^{[C]} \mid \mathscr{B}_{j,L}^{[\diamond]}) \\ & + [1 - \text{Prob}(\mathscr{B}_{j,L}^{[\diamond]})]\ \text{Prob}(\mathscr{B}_{j,L}^{[C]} \mid \text{NOT}[\mathscr{B}_{j,L}^{[\diamond]}])\end{aligned} \tag{B.15}$$

where, at this point, NOT$[\mathscr{B}_{j,L}^{[\diamond]}]$ is the only available notation for the complement of the event $\mathscr{B}_{j,L}^{[\diamond]}$.

Obviously, $\mathscr{B}_{j,L}^{[B]}$ and $\mathscr{B}_{j,L}^{[C]}$ are subsets of $\mathscr{B}_{j,L}^{[\diamond]}$, but now other connections are deemed to be possible. Nevertheless, we claim that for any $\zeta' \leqslant \zeta$, if L is sufficiently large, $\text{Prob}(\mathscr{B}_{j,L}^{[\diamond]}) \leqslant e^{-\zeta' L/3}$. Indeed, picking an $\tilde{L}$ with $1 \ll \tilde{L} \ll L$, we can define a comfort, zone around $\partial\mathbb{C}_{j,L}$ of width $\tilde{L}$ and do an expansion along the previous lines using the shapes $\mathbb{C}_{j,\tilde{L}}$ at any desirable value of $\tilde{T}$. The result is

$$\text{Prob}(\mathscr{B}_{j,L}^{[\diamond]}) \leqslant \sum_{K > (1/3)\,\tilde{L}/L} \mathfrak{M}_{\tilde{T}} \zeta_{\tilde{L}}^{\tilde{T}} \tag{B.16}$$

which immediately leads to the promised claim.

Now if the event $\mathscr{B}_{j,L}^{[\diamond]}$ occurs, we will (generously) estimate the probability of $\mathscr{B}_{j,L}^{[C]}$ by its probability with wired boundary conditions on $\mathfrak{C}_{j,L}$. But by the results and techniques of Lemma B.1, we know that this is of the order of $e^{-\tilde{\sigma}/3}$. Hence, for L large, the overall contribution from this term to the estimate in Eq. (B.14) can be bounded by $e^{-\zeta' L}e^{-\tilde{\sigma}' L}$, where ζ' is any number smaller than ζ and $\tilde{\sigma}'$ is any number smaller than $\tilde{\sigma}$. Needless to say, this is utterly negligible. This leaves us with

$$\text{Prob}(\mathscr{B}_{j,L}) \lesssim \zeta_{L/3}^{T'} \zeta_{L/3}^{T''} \text{Prob}(\mathscr{B}_{j,L}^{[C]} \mid \text{NOT}[\mathscr{B}_{j,L}^{[\diamond]}]) \tag{B.17}$$

where we have neglected terms on the right-hand side that are relatively exponentially small in L. Examining the final term, we see, as was the case in Lemma B.2, that the conditioning is "worse" (in the sense of FKG) than the probability of observing this event in the infinite-volume state. By standard decay of correlation arguments, the latter is easily estimated by const $\cdot L^2 e^{-mL/3}$. Thus, taking the appropriate action, we arrive at

$$\zeta \geqslant \tfrac{2}{3}\zeta + \tfrac{1}{3}m \tag{B.18}$$

which gives us the desired result. ∎

Proof of Theorem 2.7. The proof of Theorem 2.7 amounts to a rewrite of Proposition 2.5 and Theorem 2.6 armed with the technology of Proposition B.3. We will therefore present just the outliile of a proof. Let T be sufficiently large by the criterion of Proposition B.3. Rather than obtaining estimates on the quantities $e^{-S_n N^m}$, we will focus on events of the type $\mathscr{B}_{j,L}$ on the lattices $\mathfrak{C}_{j,L}$ with wired boundary conditions. The first step is the analog of Proposition 2.5, namely establishing that, in the specified regime, the surface tension is positive. We consider T independent realizations of the configurations $G_m^{(r)}$ placed together so as to form the shape $\mathfrak{C}_j$. We let $\mathfrak{F}_{m,r}(j)$ denote the event of a crack down the longitudinal direction of $\mathfrak{C}_j$ connecting the boundaries of the end boxes together with whatever transverse boxes are required to ensure that all of the inside squares are crossed by cracks in both directions. It is seen that if $m > k_0$, $\text{Prob}(\mathfrak{F}_{m,r}(j))$ obeys a bound analogous to the one in Eq. (2.10).

Given that $\mathfrak{F}_{m,r}(j)$ has occurred, on expanding to configurations of the type $G_{m;\ell}^{(r)}$ in each square of $\mathfrak{C}_j$, it is seen that the cost of observing the sorts of paths which constitute $\mathscr{B}_{j,L}$ will be of the order $e^{-\sigma N^\ell}$. Paths of type (i) must be treated by a relatively primitive $T=2$ argument and those of type (ii) by expanding in clusters composed of the shapes $\mathfrak{C}_{j,N^\ell}$. In the latter case, we will of course need to construct a comfort zone of width $\sim N^\ell$ lattice spacings on the wired lattices $\mathfrak{C}_{j,N^m+\ell}$ to protect ourselves from the boundary conditions. In either case, the requisite decay factor

always comes from the portions of the paths that are forced to cross the cracks.

We now examine the lattices $\mathfrak{C}_j$ of scale $N^{n+\ell}$ with $n \gg m$ and define $\mathfrak{f}$-normalcy in terms of spanning clusters composed of T-strings using the shapes $\mathfrak{C}_j$ at scale $N^{m+\ell}$. Following through with the analog of remainder of Proposition 2.5 [Eq. (2.13)–(2.16)] we have established the positivity of the surface tension. Next, a supplementary argument using the primitive $2N^{m+\ell} \times N^{m+\ell}$ squares must be employed for the benefit of the paths of type (i) that creep along the boundary: Then we have achieved a starting point for the sequences $\mathfrak{f}_k$, m_k, r_k, ℓ_k and, most importantly, w_k.

The remainder of the proof requires no significant additional modifications of the previous arguments. As with the proof of Theorem 2.6, new large-scale cracks are introduced along the lines of the event $\mathfrak{F}_{m,r}(j), j = 1, 2, ..., v_T$. To estimate the probability of observing a path of type (ii) in a wired lattice where a large crack is present, we start by introducing the appropriate "comfort zones." Using the old $\mathfrak{F}_{m_k, r_k}$'s at the small scales, we find that the fraction of the minimal journey spent in the crack allows us to replace the corresponding faction of the existing w_k with a $\sigma' \sim \sigma$ yielding an improved w_k^*. Going to a much larger scale, these cracks now become the largest scale in a network of fissures. In order to prevent the new estimate on the surface tension from sagging, in the regions where the largest fissures are absent, we must use all of the smaller scales. Finally, each stage of the argument must also include a (relatively easy) $T = 2$ subroutine to handle the paths of type (i). The ultimate conclusion is that the (w_k) are driven to σ, the value of the surface tension in the uniform system. As noted in the corollary to Theorem 2.6, each w_k also provides a bound for an appropriate cutoff surface tension, hence the desired result is automatically established for α^*. ∎

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